# Sequences and Series

## Pacing Guide

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* Benchmark students are achieving at or near grade level.

** Strategic students may be a year or more below grade-level, and may require additional time for intervention.
### Ongoing Assessment and Intervention

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**KEY:** SE = Student Edition  TE = Teacher’s Edition  CRF = Chapter Resource File  AR = Assessment Resources  Available online  Available on CD-ROM
Supporting the Teacher

Chapter 12 Resource File

- Parent Letter  
  pp. 1–2
- Practice A, B, C  
  pp. 3–5, 11–13, 19–21, 27–29, 35–37
- Review for Mastery  
- Challenge  
  pp. 8, 16, 24, 32, 40
- Problem Solving  
  pp. 9, 17, 25, 33, 41
- Reading Strategies  
  pp. 10, 18, 26, 34, 42

Transparencies

- Lesson Transparencies, Volume 4  
  Chapter 12
  - Teacher Tools
  - Warm-ups
  - Teaching Transparencies
  - Lesson Quizzes
- Alternate Openers: Explorations  
  pp. 84–88
- Know-It Notebook  
  Chapter 12
  - Vocabulary
  - Key Concepts
  - Graphic Organizers
  - Chapter Review
  - Big Ideas

Teacher Tools

- Power Presentations  
  Complete PowerPoint® presentations for Chapter 12 lessons
- Lesson Tutorial Videos  
  Holt authors Ed Burger and Freddie Renfro present tutorials to support the Chapter 12 lessons.
- Teacher’s One-Stop Planner  
  Easy access to all Chapter 12 resources and assessments, as well as software for lesson planning, test generation, and puzzle creation
- IDEA Works!  
  Key Chapter 12 resources and assessments modified to address special learning needs
- Solutions Key  
  Chapter 12
- Interactive Answers and Solutions
- TechKeys
- Project Teacher Support
- Lab Resources
- Virtual File Cabinet
- Parent Resources

Workbooks

- Homework and Practice Workbook  
  Teacher’s Edition  
  pp. 84–88
- Know-It Notebook  
  Chapter 12
- Review for Mastery Workbook  
  Teacher’s Guide  
  Chapter 12
- Focus on CAHSEE Standards: Intervention Workbook  
  Teacher’s Guide

Technology Highlights for the Teacher

- Power Presentations  
  Dynamic presentations to engage students. Complete PowerPoint® presentations for every lesson in Chapter 12.
- One-Stop Planner  
  Easy access to Chapter 12 resources and assessments. Includes lesson planning, test generation, and puzzle creation software.
- Premier Online Edition  
  Includes Tutorial Videos, Lesson Activities, Lesson Quizzes, Homework Help, Chapter Project and more.

KEY:  
- SE = Student Edition  
- TE = Teacher’s Edition  
- ELL = English Language Learners  
- SPANISH = Spanish available  
- Available online  
- Available on CD-ROM
Universal Access

Teaching Tips to help all students appear throughout the chapter. A few that target specific students are highlighted below.

**Strategic Students**
- Practice A .................................................. CRF, every lesson
- Review for Mastery ....................................... CRF, every lesson
- Reading Strategies ........................................ CRF, every lesson
- Academic Vocabulary Connections .................... TE p. 859
- Visual Cues ................................................ TE pp. 872, 881
- Questioning Strategies ..................................... TE, every example
- Ready to Go On? Intervention .......................... Chapter 12
- Know-It Notebook ....................................... Chapter 12

**English Learners**
- Reading Strategies ....................................... CRF, every lesson
- Vocabulary Exercises .................................... SE, every exercise set
- Academic Vocabulary Connections .................... TE p. 859
- English Language Learners .............................. TE pp. 861, 871, 894, 906, 922–923
- Summary and Review SPANISH ......................... Chapter 12
- Success for English Language Learners ............... Chapter 12
- Know-It Notebook ....................................... Chapter 12
- Multilingual Glossary ..................................
- Lesson Tutorial Videos ..................................

**Benchmark Students**
- Practice B .................................................. CRF, every lesson
- Problem Solving ........................................... CRF, every lesson
- Academic Vocabulary Connections .................... TE p. 859
- Questioning Strategies ..................................... TE, every example
- Ready to Go On? Intervention .......................... Chapter 12
- Know-It Notebook ....................................... Chapter 12
- Homework Help Online ..................................
- Online Interactivities ....................................

**Special Needs Students**
- Practice A .................................................. CRF, every lesson
- Review for Mastery ....................................... CRF, every lesson
- Reading Strategies ........................................ CRF, every lesson
- Academic Vocabulary Connections .................... TE p. 859
- Inclusion .................................................... TE pp. 891, 894
- IDEA Works! Modified Resources ...................... Chapter 12
- Ready to Go On? Intervention .......................... Chapter 12
- Know-It Notebook ....................................... Chapter 12
- Lesson Tutorial Videos ..................................
- Online Interactivities ....................................

**Advanced Students**
- Practice C .................................................. CRF, every lesson
- Challenge .................................................... CRF, every lesson
- Reading and Writing Math EXTENSION ............. TE p. 861
- Concept Connection EXTENSION ..................... TE pp. 888, 908
- Advanced Learners/GATE .............................. TE p. 901
- Ready to Go On? Enrichment .......................... Chapter 12

**Technology Highlights for Universal Access**

- **Lesson Tutorial Videos**
  Starring Holt authors Ed Burger and Freddie Renfro! Live tutorials to support every lesson in Chapter 12.

- **Multilingual Glossary**
  Searchable glossary includes definitions in English, Spanish, Vietnamese, Chinese, Hmong, Korean, and other languages.

- **Online Interactivities**
  Interactive tutorials provide visually engaging alternative opportunities to learn concepts and master skills.

**KEY:**
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858D
### Assessing Prior Knowledge
Determine whether students have the prerequisite concepts and skills for success in Chapter 12.

- **Are You Ready?** .......................................................... SE p. 859
- **Warm Up** ................................................................. TE, every lesson

### Chapter and Standards Assessment
Provide review and practice for Chapter 12 and standards mastery.

- **Concept Connection** .................................................. SE pp. 888, 908
- **Study Guide: Review** ............................................... SE pp. 912–915
- **Standardized Test Prep** .............................................. SE pp. 920–921
- **College Entrance Exam Practice** .................. SE p. 917

**Focus on CAHSEE Standards: Benchmark Tests**

**Focus on CAHSEE Standards: Intervention Workbook**

**California Standards Practice CD-ROM**

**CAHSEE Standards Practice CD-ROM**

**IDEA Works! Modified Worksheets and Tests**

### Alternative Assessment
Assess students’ understanding of Chapter 12 concepts and combined problem-solving skills.

- **Alternative Assessment** ........................................... TE, every lesson
- **Performance Assessment** ........................................... AR pp. 239–240
- **Portfolio Assessment** ................................................ AR p. xxxiv
- **Chapter 12 Project** ..................................................

### Daily Assessment
Provide formative assessment for each day of Chapter 12.

- **Questioning Strategies** ........................................... TE, every example
- **Think and Discuss** .................................................. SE, every lesson
- **Check It Out! Exercises** ........................................... SE, every example
- **Write About It** ....................................................... SE, every lesson
- **Journal** ................................................................. TE, every lesson
- **Lesson Quiz** .......................................................... TE, every lesson
- **Alternative Assessment** ........................................... TE, every lesson

**IDEA Works! Modified Lesson Quizzes** .................. Chapter 12

### Weekly Assessment
Provide formative assessment for each week of Chapter 12.

- **Concept Connection** ................................................ SE pp. 888, 908
- **Ready to Go On?** ..................................................... SE pp. 889, 909
- **Cumulative Assessment** ........................................... SE pp. 920–921
- **Test and Practice Generator** ................................. One-Stop Planner

### Formal Assessment
Provide summative assessment of Chapter 12 mastery.

- **Section Quizzes** ..................................................... AR pp. 225–226
- **Chapter 12 Test** ...................................................... SE p. 916
- **Chapter Test (Levels A, B, C)** .................................. AR pp. 227–238
  - Multiple Choice • Free Response
- **Cumulative Test** ................................................... AR pp. 241–244
- **Test and Practice Generator** ................................. One-Stop Planner

### Technology Highlights for Ongoing Assessment
- **Are You Ready?** ........................................................ SPANISH
- **Ready to Go On?** ....................................................... SPANISH
- **Focus on CAHSEE Standards: Benchmark Tests and Intervention**
  Automatically assess proficiency with California Algebra 2 Standards and provide intervention.
Three levels (A, B, C) of multiple-choice and free-response chapter tests are available in the Assessment Resources.

**B Chapter 12 Test**

1. **MULTIPLE CHOICE**
   - Choose the best answer.
   - 1. Which of the following functions has a horizontal asymptote of $y = 2$ and a vertical asymptote of $x = -3$?
   - 2. In which quadrant are both $x$ and $y$ negative?
   - 3. Which of the following statements is true about the graph of $y = x^2 - 4x + 3$?
   - 4. Which of the following is the equation of a line with a slope of 2 and a y-intercept of 3?
   - 5. Which of the following is the equation of a circle with center $(0, 0)$ and radius 5?
   - 6. Which of the following is the equation of a parabola that opens upward and has its vertex at $(0, 0)$?
   - 7. Which of the following is the equation of a line that is parallel to the line $y = 2x + 3$ and goes through the point $(1, 5)$?
   - 8. Which of the following is the equation of a circle that goes through the point $(1, 1)$ and has its center at $(0, 0)$?
   - 9. Which of the following is the equation of a line that is perpendicular to the line $y = -x + 2$ and goes through the point $(2, 3)$?
   - 10. Which of the following is the equation of a line that is parallel to the line $y = 3x - 2$ and goes through the point $(-1, 5)$?

2. **FREE RESPONSE**
   - 1. Given the function $f(x) = 2x^2 - 3x + 1$, find the vertex of the graph of this function.
   - 2. Find the equation of the line that goes through the points $(1, 2)$ and $(3, 4)$.
   - 3. Find the area of the region bounded by the curves $y = x^2$ and $y = 2x - 3$.

**C Chapter 12 Test**

1. **MULTIPLE CHOICE**
   - Choose the best answer.
   - 1. Which of the following functions has a horizontal asymptote of $y = 2$ and a vertical asymptote of $x = -3$?
   - 2. In which quadrant are both $x$ and $y$ negative?
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**MODIFIED FOR IDEA**

**Chapter 12 Test**

1. **MULTIPLE CHOICE**
   - Choose the best answer.
   - 1. Which of the following functions has a horizontal asymptote of $y = 2$ and a vertical asymptote of $x = -3$?
   - 2. In which quadrant are both $x$ and $y$ negative?
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2. **FREE RESPONSE**
   - 1. Given the function $f(x) = 2x^2 - 3x + 1$, find the vertex of the graph of this function.
   - 2. Find the equation of the line that goes through the points $(1, 2)$ and $(3, 4)$.
   - 3. Find the area of the region bounded by the curves $y = x^2$ and $y = 2x - 3$.

**Test & Practice Generator**

Teacher's One-Stop Planner

Create and customize Chapter 12 Tests. Instantly generate multiple test versions, answer keys, and practice versions of test items.
Sequences and Series

12A Exploring Arithmetic Sequences and Series

On page 888, students use arithmetic sequences and series to analyze tetrahedral kites. Exercises designed to prepare students for success on the Concept Connection can be found on pages 867, 876, and 886.

12B Exploring Geometric Sequences and Series

On page 908, students use geometric sequences and series to analyze and predict the box office revenue of hit movies. Exercises designed to prepare students for success on the Concept Connection can be found on pages 897 and 906.

Algebra in California

Exploring patterns, sequences, and series is an important part of mathematics. Sequences are visible all around us. For example, the Fibonacci sequence can be seen in the spirals created by the seeds in a sunflower. Students can explore the Fibonacci sequence in Lesson 12-1 and apply it to geometric forms in the Chapter Project.

About the Project

Golden Rectangles

In the Chapter Project, students explore the relationship between the Fibonacci sequence and the golden ratio. Then they construct a golden rectangle and discover how its proportions relate to the golden ratio.

Project Resources

All project resources for teachers and students are provided online.

Materials:
- calculator
- compass and straightedge (MK)
**Vocabulary**

Match each term on the left with a definition on the right.

1. exponential function E
   A. a pairing in which there is exactly one output value for each input value
2. function A
3. linear equation B
4. quadratic function D

- B. an equation whose graph is a straight line
- C. a function defined by a quotient of two polynomials
- D. a function of the form \( f(x) = ax^2 + bx + c \), where \( a \neq 0 \)
- E. a function of the form \( f(x) = ab^x \), where \( a \neq 0 \) and \( b \neq 1 \)

**Simplify Radical Expressions**

Simplify each expression.

5. \( \sqrt{25} \cdot \sqrt{36} \)
6. \( \sqrt{121} - \sqrt{81} \)
7. \( \frac{1}{49} \)
8. \( \frac{\sqrt{16}}{\sqrt{64}} \)

**Evaluate Powers**

Evaluate.

9. \((-3)^3\) 10. \((-5)^4\) 11. \(1 - (-2)^3\) 12. \(2^2 \cdot 2^7 \cdot \frac{1}{2}\)

**Solve for a Variable**

Solve each equation for \( x \).

13. \( y = 12x - 5 \) 14. \( y = -\frac{x}{3} + 1 \) 15. \( y = -9 + x^2 \) 16. \( y = -4(x^2 - 9) \)

**Evaluate Expressions**

Evaluate each expression for \( x = 2 \), \( y = 12 \), and \( z = 24 \).

17. \( \frac{y(y + 1)}{3x} \)
18. \( z + (y - 1)x \)
19. \( y \left(\frac{x + z}{2}\right) \)
20. \( z \left(\frac{1 - y}{1 - x}\right) \)

**Counterexamples**

Possible answers given.

Find a counterexample to show that each statement is false.

21. \( n^2 = n \), where \( n \) is a real number \( 2^2 \neq 2 \)
22. \( n^3 \geq n^2 \geq n \), where \( n \) is a real number
23. \( \frac{1}{n} > \frac{1}{n^2} \), where \( n \) is a real number
24. \( \frac{2}{n} \neq \frac{n}{2} \), where \( n \) is a real number

\[ \frac{1}{1} \leq \frac{1}{1^2} \text{ or } -2 < \frac{1}{(-2)^2} \]

\[ \frac{2}{2} = \frac{2}{2} \]
Chapter 12

Unpacking the Standards

Organizer

Objective: Help students organize the new concepts they will learn in Chapter 12.

Academic Vocabulary

Connections

To become familiar with some of the vocabulary terms in the chapter, consider the following. You may refer to the chapter, the glossary, or a dictionary if you like.

1. What does the word sequence mean in everyday usage? What might a number sequence refer to?
   Sequence relates to the order in which things are arranged. A number sequence might be numbers placed in order.

2. The word finite means "having definite or definable limits." Give examples of sentences that use the word finite. Explain what a finite sequence might refer to.
   There are a finite number of possible sums on a pair of number cubes. A finite sequence has an end.

3. Using the previous definition of finite, give examples of sentences that use the word infinite. Explain what an infinite sequence might refer to.
   You can combine colors in an infinite number of ways to create a painting. An infinite sequence continues without end.

Looking Back

Previously, students
- studied sets of numbers, including natural numbers and perfect squares.
- used patterns of differences or ratios to classify data.
- graphed and evaluated linear and exponential functions.

In This Chapter

Students will study
- patterns of numbers, called sequences, and their sums, called series.
- patterns to determine whether sequences are arithmetic or geometric.
- how to write and evaluate sequences and series.

Looking Forward

Students can use these skills
- in future math classes, especially Precalculus and Calculus.
- in Physics classes to model patterns, such as the heights of bouncing objects.
- to calculate the growth of financial investments.
Writing Strategy: Write a Convincing Argument

Being able to write a convincing argument about a math concept shows that you understand the material well. You can use a four-step method to write an effective argument as shown in the response to the exercise below.

**Try This**

1. A number cube is rolled 20 times and lands on the number 3 twice. What is the fewest number of rolls needed for the experimental probability of rolling a 3 to equal the theoretical probability of rolling a 3? Explain how you got your answer.

2. Aidan has narrowed his college choices down to 9 schools. He plans to visit 3 or 4 schools before the end of his junior year. How many more ways can he visit a group of 4 schools than a group of 3 schools? Explain.
<table>
<thead>
<tr>
<th>Lesson</th>
<th>Lab Resources</th>
<th>Materials</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Lesson 12-1 Introduction to Sequences</strong>&lt;br&gt;• Find the $n$th term of a sequence.&lt;br&gt;• Write rules for sequences.&lt;br&gt;☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td><em>Algebra Lab Activities</em>&lt;br&gt;12-1 Algebra Lab</td>
<td>Optional&lt;br&gt;graphing calculator</td>
</tr>
<tr>
<td><strong>Lesson 12-2 Series and Summation Notation</strong>&lt;br&gt;• Evaluate the sum of a series expressed in sigma notation.&lt;br&gt;☑ SAT-10 □ NAEP □ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>12-2 Technology Lab Evaluate Sequences and Series</strong>&lt;br&gt;• Use a graphing calculator to generate the terms of a sequence and find the sums of series.&lt;br&gt;☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td><em>Technology Lab Activities</em>&lt;br&gt;12-2 Lab Recording Sheet</td>
<td>Required&lt;br&gt;graphing calculator</td>
</tr>
<tr>
<td><strong>Lesson 12-3 Arithmetic Sequences and Series</strong>&lt;br&gt;• Find the indicated terms of an arithmetic sequence.&lt;br&gt;• Find the sums of arithmetic series.&lt;br&gt;☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td></td>
<td>Optional&lt;br&gt;centimeter squares or cubes (MK), graphing calculator</td>
</tr>
</tbody>
</table>

*MK = Manipulatives Kit*
**SEQUENCES**

**Lesson 12-1**

A **sequence** can be defined as a function \( a \) whose domain is the natural numbers (or a subset thereof) and whose range is another set of numbers. We often represent \( a(n) \) by \( a_n \) and write the sequence as an ordered list:

- \( a_1, a_2, a_3, \ldots \) (for infinite sequences)
- \( a_1, a_2, a_3, \ldots, a_n \) (for finite sequences)

Consider the sequence \( a_n = \frac{1}{2}n^2 \), which begins \( \frac{1}{2}, 2, 4, \frac{1}{2}, 8, \ldots \). As a function, the sequence maps 1 to \( \frac{1}{2} \), 2 to 2, 3 to \( 4, \frac{1}{2} \), and 4 to 8. These points are plotted on the coordinate plane below. The graph shows that the points lie on the curve \( y = \frac{1}{2}x^2 \). In other words, the sequence \( a_n = \frac{1}{2}n^2 \) is precisely the function \( y = \frac{1}{2}x^2 \) with its domain restricted to the natural numbers.

However, it is important to understand that many sequences are not given by simple algebraic formulas and in fact may appear to be just random sequences of digits.

**SERIES**

**Lesson 12-2**

A **series** is the sum of the terms of a sequence. A series may add a finite number of terms or an infinite number of terms. A partial sum, \( S_n \), is the sum of the first \( n \) terms of a sequence.

A well-known story about the mathematician Carl Friedrich Gauss (1777–1855) states that a harried teacher once asked Gauss and his classmates to add all of the natural numbers from 1 to 100. To the teacher’s surprise, this task did not keep all the students quiet for very long; Gauss immediately gave the correct answer, 5050. He saw that the terms of the series can be rearranged to form 50 pairs that add to 101:

\[
1 + 100 = 101 \\
2 + 99 = 101 \\
3 + 98 = 101 \\
\vdots \\
50 + 51 = 101
\]

This is the key idea behind the formula for the sum of a linear series:

\[
\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}
\]

**ARITHMETIC SEQUENCES AND SERIES**

**Lesson 12-3**

In an **arithmetic sequence**, successive terms differ by the same nonzero number \( d \), called the **constant difference**. Because the \( n \)th term can be found by adding \( d(n - 1) \) times to the first term, the general rule for the \( n \)th term of an arithmetic sequence is

\[
a_n = a_1 + (n - 1)d.
\]

An arithmetic sequence can be viewed as a linear function whose domain is restricted to the natural numbers. In fact, the general rule above has much in common with the equation of a line in slope-intercept form, \( y = mx + b \). In particular, the constant difference \( d \) is the slope of the line through the points \((n, a_n)\).

The sum of the first \( n \) terms of an arithmetic series \( S_n \) is given by the formula

\[
S_n = n \left( \frac{a_1 + a_n}{2} \right).
\]

This formula may be developed as follows. The sum of the series \( S_n \) divided by the number of terms \( n \) is the average of the terms. The average of the terms must be \( \frac{a_1 + a_n}{2} \) since the terms are equally spaced and symmetric about this value on a number line. Thus,

\[
\frac{S_n}{n} = \frac{a_1 + a_n}{2},
\]

which immediately yields the required formula.
Introduction to Sequences

Who uses this?

Sequences can be used to model many natural phenomena, such as the changes in a rabbit population over time.

In 1202, Italian mathematician Leonardo Fibonacci described how fast rabbits breed under ideal circumstances. Fibonacci noted the number of pairs of rabbits each month and formed a famous pattern called the Fibonacci sequence.

A sequence is an ordered set of numbers. Each number in the sequence is a term of the sequence. A sequence may be an infinite sequence that continues without end, such as the natural numbers, or a finite sequence that has a limited number of terms, such as \( \{1, 2, 3, 4, 5\} \).

You can think of a sequence as a function with sequential natural numbers as the domain and the terms of the sequence as the range. Values in the domain are called term numbers and are represented by \( n \). Instead of function notation, such as \( a(n) \), sequence values are written by using subscripts. The first term is \( a_1 \), the second term is \( a_2 \), and the \( n \)th term is \( a_n \). Because a sequence is a function, each number \( n \) has only one term value associated with it, \( a_n \).

A recursive formula is a rule in which one or more previous terms are used to generate the next term.

**Example 1**

Find the first 5 terms of a sequence by using a recursive formula.

Find the first 5 terms of the sequence with \( a_1 = 5 \) and 
\[ a_n = 2a_{n-1} + 1 \] for \( n \geq 2 \).

The first term is given, \( a_1 = 5 \).

**Substitute** \( a_1 \) **into the rule to find** \( a_2 \).

**Continue using** each term to find the next term.

The first 5 terms are 5, 11, 23, 47, and 95.

Find the first 5 terms of each sequence.

1a. \( a_1 = -5 \), \( a_n = a_{n-1} - 8 \)
2b. \( a_1 = 2 \), \( a_n = -3a_{n-1} \)

-5, -13, -21, -29, -37
2, -6, 18, -14, 28

**Motivate**

Ask students to identify the next 3 terms in the pattern 2, 4, 6, 8, . . . . 10, 12, 14 Discuss how they decided on their answers. Then ask students to identify the next 3 terms in the pattern 2, 4, 8, 16, . . . . 32, 64, 128 Discuss how they decided on their answers. Explain that number patterns like these that follow a rule are called sequences.

Explorations and answers are provided in Alternate Openers: Explorations Transparencies.
In some sequences, you can find the value of a term when you do not know its preceding term. An explicit formula defines the nth term of a sequence as a function of n.

**EXAMPLE 2** Finding Terms of a Sequence by Using an Explicit Formula

Find the first 5 terms of the sequence \( a_n = 2^n - 3 \).

Make a table. Evaluate the sequence for \( n = 1 \) through \( n = 5 \).

The first 5 terms are \(-1, 1, 5, 13, \) and \(29\).

Check Use a graphing calculator. Enter \( y = 2^x - 3 \) and make a table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2^n - 3 )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 2^1 - 3 )</td>
<td>(-1)</td>
</tr>
<tr>
<td>2</td>
<td>( 2^2 - 3 )</td>
<td>(1)</td>
</tr>
<tr>
<td>3</td>
<td>( 2^3 - 3 )</td>
<td>(5)</td>
</tr>
<tr>
<td>4</td>
<td>( 2^4 - 3 )</td>
<td>(13)</td>
</tr>
<tr>
<td>5</td>
<td>( 2^5 - 3 )</td>
<td>(29)</td>
</tr>
</tbody>
</table>

Find the first 5 terms of each sequence.

2a. \( a_n = n^2 - 2n - 1 \), \(0, 3, 8, 15\)

2b. \( a_n = 3n - 5 \), \(-2, 1, 4, 7, 10\)

You can use your knowledge of functions to write rules for sequences.

**EXAMPLE 3** Writing Rules for Sequences

Write a possible explicit rule for the nth term of each sequence.

A. \(3, 6, 12, 24, 48, \ldots\)

Examine the differences and ratios.

<table>
<thead>
<tr>
<th>Terms</th>
<th>3</th>
<th>6</th>
<th>12</th>
<th>24</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st differences</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>2nd differences</td>
<td>3</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The ratio is constant. The sequence is exponential with a base of 2.

\( a_1 = 3(2)^0, a_2 = 6(2)^1, a_3 = 12(2)^2, \ldots \)

A pattern is \(3(2)^{n-1}\). One explicit rule is \( a_n = 3(2)^{n-1} \).

B. \(2, 5, 4, 5.5, 7, 8.5, \ldots\)

Examine the differences.

<table>
<thead>
<tr>
<th>Terms</th>
<th>2.5</th>
<th>4</th>
<th>5.5</th>
<th>7</th>
<th>8.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st differences</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The first differences are constant, so the sequence is linear.

The first term is 2.5, and each term is 1.5 more than the previous.

A pattern is \(2.5 + 1.5(n - 1)\), or \(1.5n + 1\). One explicit rule is \( a_n = 1.5n + 1 \).

Write a possible explicit rule for the nth term of each sequence.

3a. \(7, 5, 3, 1, -1, \ldots\) \( a_n = 9 - 2n \)

3b. \(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\) \( a_n = \frac{1}{n} \)

**INTERVENTION Questioning Strategies**

**EXAMPLE 1**

- Is the sequence increasing or decreasing?
- What effect would a negative multiplier have on the sequence? Explain.

**EXAMPLE 2**

- How is using an explicit formula different from using a recursive formula?

**EXAMPLE 3**

- How do differences or ratios help identify the explicit rule for the nth term?

**Universal Access Through Cognitive Strategies**

Students may need help with the new notation. Stress that \( n \) is the term number (also called the *index of a term*) and \( a_n \) is the nth term, so \( a_{n-1} \) is the previous term and \( a_{n+1} \) is the next term. To reinforce these concepts, have students identify some terms for a given \( n \)-value.

For example, which term is \( a_{n+2} \) when \( n = 3? \)

**Teach**

**Guided Instruction**

Compare sequences with the functions students have already learned. Choose a simple linear function, such as \( f(x) = 5x \).

Have students evaluate the function for several consecutive integer values. Ask them to describe the results. counting by 5’s or multiples of 5. Use the answer to explain the difference between recursive and explicit formulas.
A ball is dropped and bounces to a height of 5 feet. The ball rebounds to 60% of its previous height after each bounce. Graph the sequence and describe its pattern. How high does the ball bounce on its 9th bounce?

Because the ball first bounces to a height of 5 feet and then bounces to 60% of its previous height on each bounce, the recursive rule is \( a_1 = 5 \) and \( a_n = 0.6a_{n-1} \). Use this rule to find some other terms of the sequence and graph them.

\[
\begin{align*}
  a_2 &= 0.6(5) = 3 \\
  a_3 &= 0.6(3) = 1.8 \\
  a_4 &= 0.6(1.8) = 1.08
\end{align*}
\]

The graph appears to be exponential. Use the pattern to write an explicit rule.

\[
a_n = 5(0.6)^{n-1}, \text{ where } n \text{ is the bounce number}
\]

Use this rule to find the bounce height for the 9th bounce.\( a_9 = 5(0.6)^{9-1} = 0.084 \text{ foot, or approximately 1 inch.} \)

The ball is about 0.084 feet high on the 9th bounce.

4. An ultra-low-flush toilet uses 1.6 gallons every time it is flushed. Graph the sequence of total water used after \( n \) flushes, and describe its pattern. How many gallons have been used after 10 flushes?

The graph shows the points lie on a line with positive slope; 16 gal.

Recall that a fractal is an image made by repeating a pattern (Lesson 5-5). Each step in this repeated process is an iteration, the repetitive application of the same rule.

\[
\begin{align*}
  a_1 &= 1 \\
  a_2 &= 3 \\
  a_3 &= 9 \\
  a_4 &= 27 \\
  a_5 &= 81
\end{align*}
\]

The Sierpinski triangle is a fractal made by removing the center of each triangle, each iteration repeating for each new triangle. Find the number of triangles in the next 2 iterations.

By removing the center of each triangle, each iteration turns every triangle into 3 smaller triangles. So the number of triangles triples with each iteration.

The number of triangles can be represented by the sequence \( a_n = 3^{n-1} \).

The 4th and 5th terms are \( a_4 = 3^{4-1} = 27 \) and \( a_5 = 3^{5-1} = 81 \).

The next two iterations result in 27 and 81 triangles.

5. The Cantor set is a fractal formed by repeatedly removing the middle third of a line segment as shown. Find the number of segments in the next 2 iterations. \( 8, 16 \)

---

**Example 1**

A ball is dropped and bounces to a height of 4 feet. The ball rebounds to 70% of its previous height after each bounce. Graph the sequence and describe its pattern. How high does the ball bounce on its 10th bounce?

\[
\begin{align*}
  a_n &= 4(0.7)^{n-1}; \approx 0.161 \text{ ft, or } \approx 2 \text{ in.}
\end{align*}
\]

**Example 5**

Find the number of triangles in the 7th and 8th iterations of the Sierpinski triangle. \( 729, 2187 \)

Also available on transparency

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**Close**

**Summarize**

Ask students to define a sequence. A sequence is an ordered set of numbers. Then have them explain how to use a recursive formula and an explicit formula to find the terms of a sequence. A recursive formula uses previous terms to generate successive terms. An explicit formula defines the \( n \)th term of a sequence as a function of \( n \).
1. Explain the difference between a recursive rule and an explicit rule.
2. Identify three possible next terms for the sequence 1, 2, 4, ...
3. Describe how a sequence is a function. Do all sequences have the same domain? Explain.
4. GET ORGANIZED Copy and complete the graphic organizer. Summarize what you have learned about sequences.

500, 100, 20, 4, 5

1. a(n) = a(n-1) + 3
2. a(n) = 2n - 1
3. a(n) = 3n - 1
4. a(n) = (n + 1)^2

Find the first 5 terms of each sequence.
1. a1 = 1, a2 = 4a1 - 1, a3 = 4a2 - 1
2. a1 = 3, a2 = a1 + 11
3. a1 = 500, a2 = \( \frac{a_1}{5} \)
4. a1 = -3n^2
5. a(n) = n(n - 1)
6. a(n) = 4^{n-1}
7. a(n) = (n + 1)^2
8. a(n) = 2^n
9. a(n) = 3^n
10. a(n) = 3 + 3n
11. a(n) = 3 + 3n
12. a(n) = 3 + 3n
13. a(n) = (n + 1)^2
14. a(n) = 3^n
15. a(n) = \( \frac{n}{n+1} \)
16. a(n) = \( \frac{n}{n+1} \)
17. a(n) = \( \frac{n}{n+1} \)
18. a(n) = 1.5a(n-1) - 2
19. a(n) = (2^n) - 8
20. a(n) = 2n^2 - 12
21. a(n) = -2, a(n) = -3a(n-1) - 1

There is only one output, \( a_n \), associated with each input, \( n \). The domain of a sequence is the set of term numbers, but sequences do not necessarily have the same domain.

Possible answers:
1. A recursive rule uses one or more previous terms to find the next term. An explicit rule uses the first term and the term number to find the value.
2. 1, 2, 7
3. Answers to Think and Discuss

1. \( a(n) \) = \( \frac{n}{n+1} \)
2. \( a(n) \) = \( \frac{n}{n+1} \)
3. \( a(n) \) = \( \frac{n}{n+1} \)
4. \( a(n) \) = \( \frac{n}{n+1} \)

### Practice and Problem Solving

Find the first 5 terms of each sequence.

16. \( a_1 = 7, a_2 = a_1 - 3 \)
17. \( a_1 = 1, a_2 = \frac{1}{4}, a_3 = \frac{1}{9}, a_4 = \frac{1}{16}, a_5 = \frac{1}{25} \)
18. \( a_1 = 4, a_2 = 1.5a_1 - 2 \)
19. \( a_1 = (2^n) - 8 \)
20. \( a_1 = 2n^2 - 12 \)
21. \( a_1 = -2, a_2 = -3a_1 - 1 \)

There is only one output, \( a_n \), associated with each input, \( n \). The domain of a sequence is the set of term numbers, but sequences do not necessarily have the same domain.

Possible answers:
1. A recursive rule uses one or more previous terms to find the next term. An explicit rule uses the first term and the term number to find the value.
2. 1, 2, 7
3. Answers to Think and Discuss

### Technology

Students are taught to use the seq function on graphing calculators in the Technology Lab following Lesson 12-2. You may want to teach the first part of that lab before or during this lesson.

### Answers

5. -12, 0, 12, 24, 36
6. -1
7. -3, -12, -27, -48, -75

### Homework Quick Check

Quickly check key concepts. Exercises: 16, 20, 22, 25, 26, 28

### Assignment Guide

Assign Guided Practice exercises as necessary.

If you finished Examples 1–2
Basic 16–21, 27–29
Proficient 16–21, 27–31
Advanced 16–21, 27–32

If you finished Examples 1–5
Proficient 16–46, 49–54, 59–66
Advanced 16–33, 34–46 even, 47–66

### Answers

5. -12, 0, 12, 24, 36
6. -1
7. -3, -12, -27, -48, -75

### Answers

5. -12, 0, 12, 24, 36
6. -1
7. -3, -12, -27, -48, -75

### Answers

5. -12, 0, 12, 24, 36
6. -1
7. -3, -12, -27, -48, -75
Write a possible explicit rule for the nth term of each sequence.

### 22. 2, 8, 18, 32, 50, …
23. 9, 5, 1, −3, −7, …
24. 5, 0.5, 0.05, 0.005, …

### 25. Architecture
Chairs for an orchestra are positioned in a curved form with the conductor at the center. The front row has 16 chairs, and each successive row has 4 more chairs. Graph the sequence and describe its pattern. How many chairs are in the 6th row? Linear with a slope of 4; 36

### 26. Fractals
Find the number of squares in the next 2 iterations of Cantor dust as shown. 256, 1024

Find the first 5 terms of each sequence. 1, 1, 2, 2, 3, 3, −10, 20, −10, 20, −10
27. \(a_n = 12, a_{n+1} = \frac{2}{3}a_n + 2\)
28. \(a_n = 0, a_{n+1} = \frac{2}{3}a_n + 1\)
29. \(a_n = -10, a_{n+1} = -\frac{2}{3}a_n + 10\)
30. \(a_n = 2n^2 - 12\)
31. \(a_n = 8 - \frac{1}{n}\)
32. \(a_n = 5(-1)^{n+1}(3)^{n-1}\)

### 33. Error Analysis
Two attempts to find the first 5 terms of the sequence \(a_n = 13 - 2n + 1\) are shown. Which is incorrect? Explain. \(\mathbf{b}\) is incorrect. The formula is explicit, not recursive.

### 34. 16, 4, 1, \(\frac{1}{4}\), \(\frac{1}{16}\), …
35. 15, 14, 13, 12, 11, 9, 9, 9, 9, 9, 9, …
36. −5.0, −2.5, 0, 2.5, 5.0, 7.5, …

### 37. \(1, -\frac{1}{2}, \frac{3}{4}, -\frac{5}{8}, \frac{7}{16}, \ldots\)
38. 0.04, 0.4, 4.0, 40.0, 400.0, …

### 39. 24, 21, 16, 9, 0, …
40. Fibonacci
Recall from the lesson that the Fibonacci sequence models the number of pairs of rabbits after a certain number of months. The sequence begins 1, 1, …, and each term after that is the sum of the two terms before it.

- a. Find the first 12 terms of the Fibonacci sequence.
- b. How many pairs of rabbits are produced under ideal circumstances at the end of one year? 144 rabbit pairs

Find the first 10 terms of each sequence.

### 41. \(a_1, a_2, a_3, 15, 21\)
42. \(a_1, a_2, a_3, 25, 36\)

### 43. Chess
Ronnie is scheduling a chess tournament in which each player plays every other player once. He created a table and found that each new player added more than one game.

- a. Graph the sequence and describe its pattern. What are the next 2 terms in the sequence?
- b. Use a regressive rule for the sequence.
- c. What if…? How would the schedule change if each player played every other player twice? Make a table, and describe how the sequence is transformed.

### 12-1 Practice A

#### 12-1 Practice C

Find the first 10 terms of each sequence.

- \(a_1, a_2, a_3, \ldots\)
- \((a_n)\)
- \((b_n)\)
- \((c_n)\)
- \((d_n)\)
- \((e_n)\)

#### 12-1 Practice B

Find the first 10 terms of each sequence.

- \(a_1, a_2, a_3, \ldots\)
- \((a_n)\)
- \((b_n)\)
- \((c_n)\)
- \((d_n)\)
- \((e_n)\)

#### 12-1 Reading Strategies

**Sequence and patterns are often represented by explicit rules, such as \(a_n = n\), \(a_n = n^2\), \(a_n = 2^n\), and so forth, in which the term follows a pattern.**

- **Explicit Rule:**
  - The 4th term is 16.
  - The nth term is 4n.

**Finding the first 5 terms of the sequence.**

- \(a_1, a_2, a_3, a_4, a_5\)

**Finding the first 10 terms of the sequence.**

- \(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\)

**Finding the nth term of the sequence.**

- \(a_n\)

**Finding the value of each term.**

- \(a_{n+1}\)

**Finding the term that follows a specific term.**

- \(a_{n+1}\)

**Finding the term that precedes a specific term.**

- \(a_{n-1}\)

**Finding the term that equals a specified value.**

- \(a_n = \text{specified value}\)

**Finding the value of each term when the term number is a variable.**

- \(a_n = \text{term number}\)

**Finding the term that equals a specified value when the term number is a variable.**

- \(a_n = \text{specified value}\)

**Finding the term that equals a specified value when the term number is a variable.**

- \(a_n = \text{specified value}\)

**Finding the term that equals a specified value when the term number is a variable.**

- \(a_n = \text{specified value}\)
44. This problem will prepare you for the Concept Connection on page 888. A kite is made with 1 tetrahedron in the first (top) layer, 3 tetrahedrons in the second layer, 6 tetrahedrons in the third layer, and so on. Each tetrahedron is made by joining six sticks of equal length.

a. The rule \( a_n = a_{n-1} + 6n \) gives the number of sticks needed to make the \( n \)th layer of the kite, where \( a_1 = 6 \).

b. Find the first five terms of the sequence: 6, 18, 36, 60, 90.

c. Use regression to find an explicit rule for the sequence: \( a_n = 3n^2 + 3n \).

d. How many sticks are needed to build the 10th layer of the kite? 330

46. Possible answer: about 27; the first term is almost 8. Each term is about 1 more than the previous term and \( 8 + 19 = 27 \).

47a. 1, 2, 4, 8, 16, 32

47b. 1, 2, 4, 8, 16, 32

47. Music

Music involves arranging different pitches through time. The musical notation below indicates the duration of various notes (and rests).

Symbols for Musical Notes and Rests

Whole Half Quarter Eighth Sixteenth Thirty-second

Critical Thinking

In Exercises 45, some students may remember the formula for the sum of the interior angle measures from geometry. Encourage those students to use the method presented in the problem to verify the formula they remember.

Answers

43a. It appears to be quadratic.

45b. Sides | Angle Measure \(^{(*)}\)
---|---
3 | 60
4 | 90
5 | 108
6 | 120
7 | 128 \(\frac{1}{2}\)
8 | 135

45c. Players | 1 | 2 | 3 | 4 | 5
---|---|---|---|---|---
Games | 0 | 2 | 6 | 12 | 20

The sequence is twice the previous sequence. The function is a vertical stretch by a factor of 2.
48. **Critical Thinking** Can the recursive rule and the explicit rule for a formula ever be the same?

49. **Write About It** Explain how an infinite sequence is different from a finite sequence.

---

**Standardized Test Prep**

50. Which is the next term in the sequence \(-9, -6, -3, 0, \ldots\)?
   - A \(-3\)
   - B 0
   - C 3
   - D 6

51. Which rule describes the given sequence 4, 12, 36, 108, \ldots?
   - \(a_n = 4 + 3n\)
   - \(a_n = 4 + 3n\)
   - \(a_n = 4, a_n = 3a_{n-1}, n \geq 2\)
   - \(a_n = 4, a_n = 3a_{n-1}, n \geq 2\)

52. Which sequence is expressed by the rule \(a_n = \frac{2n}{n + 1}\)?
   - \(1, 4, 3, 8, 5, \ldots\)
   - \(0, 1, 2, 3, 8, 5, \ldots\)
   - \(2, 3, 8, 5, 12, 7, \ldots\)
   - \(2, 3, 8, 5, 12, 7, \ldots\)

53. Which sequence is expressed by the rule \(a_n = 6\) and \(a_n = 12 - 2a_{n-1}, n \geq 2\)?
   - \(6, 4, 2, 0, -2, -4, \ldots\)
   - \(6, 0, 12, -12, 36, \ldots\)
   - \(6, 0, -6, -12, -18, \ldots\)
   - \(6, 0, -6, -12, -18, \ldots\)

54. **Gridded Response** Find the next term in the sequence \(-32, 16, -8, 4, -2, \ldots\)

---

**Challenging and Extend**

55. \(a_n = \frac{n^3}{3^2 - 1}\)

56. \(a_n = (-1)^n (n^2 + n)\): 110

57. \(a_n = -0.05n^2 + 0.05n + 0.9\): -3.6

---

**Spiral Review**

Simplify. Assume that all expressions are defined. (Lesson 8-2)

59. \(\frac{x^2 - 9}{x^2 + 5x + 6}\) \(\frac{x - 3}{x + 2}\)

60. \(\frac{4x^2 - 5x}{8x^2 + 18x - 35}\) \(\frac{2x + 7}{2x - 7}\)

63. \(2(3x^2 - 5x - 3)\) \((x + 1)(x - 1)\)

64. \(8x^3 - 7\) \((2x - 1)\)

66. **Literature** Christopher is reading a book containing 854 pages at a rate of 1.5 pages per minute. Create a table, equation, and graph to represent the number of pages remaining to be read \(P\) in relation to time \(t\). (Lesson 9-1)
Sequences of figures can often be described with number patterns.

Examples of figures can often be described with number patterns.

Polygons can be represented by dots arranged in the form of polygons. The first four hexagonal numbers are illustrated. The sequence for the total number of dots is 1, 6, 15, 28, ....

Example

The figure shows how regular hexagons can be used to tessellate, or cover, the plane. Write a sequence for the number of hexagons added at each stage. Describe the pattern in the sequence, and find the next term.

Show the number of hexagons added at each stage.

<table>
<thead>
<tr>
<th>Level</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Hexagons</td>
<td>1</td>
<td>6</td>
<td>12</td>
<td>18</td>
</tr>
</tbody>
</table>

From the second term on, the number added appears to increase by 6 each time. The next level probably would have 24 hexagons. You can check your conjecture by constructing the next stage of the tessellation and counting the hexagons.

Try This

Write a sequence for the number of polygons added at each stage of the figure. Describe the pattern in the sequence, and find the next term.

1. 1, 3, 5, 7, ....; consecutive odd numbers; 9

2. 1, 8, 16, 24, ....; after the second term, each term increases by 8; 32

3. 3, 9, 15, ....; each term increases by 6; 27
Series and Summation Notation

**Objective**
Evaluate the sum of a series expressed in sigma notation.

**California Standards**
Preparation for 23.0

Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.

**Why learn this?**
You can use sums of sequences to find the size of a house of cards. (See Example 4.)

In Lesson 12-1, you learned how to find the nth term of a sequence. Often we are also interested in the sum of a certain number of terms of a sequence. A series is the indicated sum of the terms of a sequence. Some examples are shown in the table.

<table>
<thead>
<tr>
<th>Sequence</th>
<th>1, 2, 3, 4</th>
<th>2, 4, 6, 8, …</th>
<th>1 ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{4} ) ( \frac{1}{8} ) ( \frac{1}{16} ) ( \frac{1}{32} ) ( \frac{1}{64} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Series</td>
<td>1 + 2 + 3 + 4</td>
<td>2 + 4 + 6 + 8 + …</td>
<td>( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} )</td>
</tr>
</tbody>
</table>

Because many sequences are infinite and do not have defined sums, we often find partial sums. A partial sum, indicated by \( S_n \), is the sum of a specified number of terms of a sequence.

A series can also be represented by using summation notation, which uses the Greek letter \( \Sigma \) (capital sigma) to denote the sum of a sequence defined by a rule, as shown.

\[
\sum_{k=1}^{n} 2k
\]

**Example 1**

Using Summation Notation

Write each series in summation notation.

\( A \)

2. Write the notation for the first 5 terms.

\[
\sum_{k=1}^{5} 3k
\]

\( B \)

2. Write the notation for the first 6 terms.

\[
\sum_{k=1}^{6} \left(-1\right)^{k+1} \left(\frac{1}{2}\right)^k
\]

Motivate
Ask students if they have ever received any chain e-mails. Have them imagine that 2 people receive a chain e-mail on Monday, and each of those sends it to 2 people on Tuesday, and so on. How many people have received the e-mail at the end of one week? Explain that the answer to this question is the sum of a sequence, or series.

Explorations and answers are provided in Alternate Openers: Explorations Transparencies.
Guided Instruction

Define a series and explain the relationship to a sequence. Review summation notation, and ask students to clarify what each variable or expression represents. Reinforce the notation verbally and visually because this material may be new for most students. Point out that the summation formulas on p. 871 are convenient ways to find the sums of common series.

Universal Access

Through Cooperative Learning

Have groups of students create a sequence and a related series to model a real-world situation. Have them generate a question that draws on the terms of a sequence and a question that requires the sum of the related series. For example, cans are stacked in a pyramidal display with 1 in the top row, 2 in the 2nd row, 3 in the 3rd row, etc. How many cans are in the 5th row? sequence. How many total cans are in a display with 10 rows? series

Additional Examples

Example 1

Write each series in summation notation.

A. \(4 + 8 + 12 + 16 + 20\)

\[\sum_{k=1}^{5} (4k)\]

B. \(-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36}\)

\[\sum_{k=1}^{6} (-1)^{k+1} \left(\frac{1}{k^2}\right)\]

Example 2

Expand each series and evaluate.

A. \(\sum_{k=1}^{5} \frac{1}{3^k}\) \(= \frac{\frac{1}{3} - \frac{1}{3^6}}{1 - \frac{1}{3}}\) = \(\frac{40}{243}\)

B. \(\sum_{k=1}^{5} (k^2 - 10) (1^2 - 10) + (4^2 - 10) + (5^2 - 10) + (6^2 - 10)\)

\[= 31\]

Also available on transparency

INTERVENTION

Questioning Strategies

EXAMPLE 1

• How do you use the terms of the series to help find the rule for the series?

• How do you know what to write for \(k\) in summation notation?

EXAMPLE 2

• How do you know the starting and ending values when you expand and evaluate a series?

• How does expanding a series differ from evaluating the series?
Evaluate each series.

A. \( \sum_{k=1}^{10} k = 42 \)

B. \( \sum_{k=1}^{12} k = 78 \)

C. \( \sum_{k=1}^{12} k^2 = 650 \)

Example 4
Sam is laying out patio stones in a triangular pattern. The first row has 2 stones and each row has 2 additional stones, as shown below. How many complete rows can he make with a box of 144 stones? 11

Row 1
Row 2
Row 3

Also available on transparency

INTERVENTION

Questioning Strategies

EXAMPLE 3
• How do you recognize a constant, linear, or quadratic series?
• Are there types of series other than constant, linear, or quadratic? How do you know?

EXAMPLE 4
• How does a table or diagram help you identify a pattern?
• Why do you need to identify a pattern at all?
• How does the series represent the total?

Chapter 12 Sequences and Series

Universal Access
Through Visual Cues

Have students create a poster of summation notation to display in the classroom. Have them use color coding to indicate that \( k \) is the value that changes with each new term in a series.
THINK AND DISCUSS

1. Explain the difference between a sequence and a series.

2. Explain what each of the variables represents in the notation \( \sum_{k=m}^{n} k \).

3. GET ORGANIZED Copy and complete the graphic organizer. Write the general notation and an example for each term.

Answers to Think and Discuss

Possible answers:

1. A sequence is an ordered list of terms. When the terms are added together, a series is created. The series is an indicated sum of the sequence.

2. \( m \) is the first replacement for \( k \), \( n \) is the last replacement for \( k \), and \( k \) is the explicit formula.

1. **Vocabulary** Give an example of summation notation. \( \sum_{k=1}^{n} k \)

2. **SEE EXAMPLE p. 870** Write each series in summation notation.
   - 2. \( 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \)
   - 3. \( -3 + 6 - 9 + 12 - 15 \)
   - 4. \( 1 + 10 + 100 + 1000 + 10,000 \)
   - 5. \( 1,000 + 95 + 90 + 85 + 80 \)

3. **SEE EXAMPLE p. 871** Expand each series and evaluate.
   - 6. \( \sum_{k=1}^{5} k \)
   - 7. \( \sum_{k=1}^{5} (k+1) \)
   - 8. \( \sum_{k=1}^{5} -k \)

4. **SEE EXAMPLE p. 872** Evaluate each series.
   - 9. \( \sum_{k=1}^{11} k \)
   - 10. \( \sum_{k=1}^{20} k^2 \)
   - 11. \( \sum_{k=1}^{6} 12 \)

5. **SEE EXAMPLE p. 872**

6. **Finance** Melinda makes monthly car payments of $285 each month. How much will she have paid after 2 years? 5 years? $\$6840; \$17,100

7. **Practice and Problem Solving** Write each series in summation notation.
   - 13. \( 1.1 + 2.2 + 3.3 + 4.4 + 5.5 \)
   - 14. \( \frac{1}{2} + \frac{1}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} \)
   - 15. \( 1 + 2 + 3 + 4 + 5 + 6 \)

8. **Independent Practice**
   - 13–16 1
   - 17–19 2
   - 20–22 3
   - 23 4

9. **Extra Practice**

   - 16. \( \sum_{k=1}^{10} (k+3) \)
   - 17. \( \sum_{k=1}^{10} (-k) \)
   - 18. \( \sum_{k=1}^{4} (k - 1) \)

10. **Travel** The distance from St. Louis, Missouri, to Los Angeles, California, is 1596 miles. Michael plans to travel half the distance on the first day and half the remaining distance each day after that. Write a series in summation notation for the total distance he will travel in 5 days. How far will Michael travel in the 5 days? $\sum_{k=1}^{5} 1596 \frac{1}{2}$; 1546 $\frac{1}{8}$ mi

---

**12-2 Practice A**

1. **Evaluate each series.**
   - A. \( 1 + 2 + 3 + 4 + 5 \)
   - B. \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \)

2. **Write the series in compact notation using summation notation.**
   - A. \( 1 + 2 + 3 + 4 + 5 \)
   - B. \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \)

---

**12-2 Practice B**

1. **Write each series in expanded notation.**
   - A. \( 1 + 2 + 3 + 4 + 5 \)
   - B. \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \)

2. **Expand each series and evaluate.**
   - A. \( 1 + 2 + 3 + 4 + 5 \)
   - B. \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \)

---

**Key Words:**

- Sequences
- Series
- Summation notation
- Expand each series and evaluate
- Evaluate each series
- Practice and problem solving
- Independent practice
- Extra practice
35. **Safety** An employer uses a telephone tree to notify employees in the case of an emergency closing. When the office manager makes the decision to close, she calls 3 people. Each of these people calls 3 other people, and so on.

a. Make a tree diagram with 3 levels to represent the problem situation.
b. Write and evaluate a series to find the total number of people notified after 5 levels of calls.
c. **What if...?** Suppose that the phone tree involves calling 5 people at each level. How many more people would be notified after 5 levels of calls? 3542

42. **Architecture** A hotel is being built in the shape of a pyramid, as shown in the diagram. Each square floor is 10 feet longer and 10 feet wider than the floor above it.

a. Write a series that represents the total area of \( n \) floors of the hotel.
b. How many stories must the hotel be to have at least 50,000 square feet of floor area?

---

42a. \[ \sum_{k=1}^{n} 100(k + 1)^2 \]

**Estimation** Use mental math to estimate each sum. Then compare your answer to the sum obtained by using a calculator.

43. \[ 10 + 11 + 12 + \cdots + 29 + 30 \]
44. \[ 1 + 3 + 5 + \cdots + 97 + 99 \]
45. \[ -2 + (-4) + (-6) + \cdots + (-100) \]

46. **Physics** The distance that an object falls in equal time intervals is represented in the table. Rules created by Leonardo da Vinci and Galileo are shown. (In this general case, specific units do not apply to time or distance.)

a. Write the series for each rule for 5 intervals, and find the respective sums. What does the sum of the series for 5 intervals represent?
b. Write each series in sumation notation. Then evaluate each series for \( n = 10 \).
c. By the current rule, the distance fallen in each interval is 1, 4, 9, 16, 25, ...

47. **Critical Thinking** Some mathematical properties may be applied to series.

a. Evaluate \( \sum_{k=1}^{10} \frac{k}{k} \) and \( 3 \sum_{k=1}^{10} \). Make a conjecture based on your answer.
b. Evaluate \( \sum_{k=1}^{10} k + \sum_{k=1}^{10} 2 \) and \( \sum_{k=1}^{10} (n + 2) \). Make a conjecture based on your answer.

---

### 12-2 Series and Summation Notation

#### Exercise 46a.

Leonardo: \[ 1 + 2 + 3 + 4 + 5 = 15 \]

Galileo: \[ 1 + 3 + 5 + 7 + 9 = 25 \]

The total distance traveled in 5 equal units of time

b. Leonardo: \[ \sum_{k=1}^{5} k; 55 \]

Galileo: \[ \sum_{k=1}^{5} (2k - 1); 100 \]

c. Objects fall farther under the current rule.

### 47a. Both equal 165;

\[ \sum_{k=1}^{n} c a_k = \sum_{k=1}^{n} a_k \]

b. Both equal 75;

\[ \sum_{k=1}^{n} \left( a_k + b_k \right) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \]
48. This problem will prepare you for the Concept Connection on page 888.

The series $\sum_{k=1}^{n} (3k^2 + 3k)$ gives the total number of sticks needed to make a tetrahedral kite with $n$ layers.

a. Expand and evaluate the series to find out how many sticks are needed to make a kite with 5 layers.

b. Use the properties $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ and $\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$ to rewrite the series as a multiple of a quadratic series plus a multiple of a linear series.

c. Use summation formulas to determine how many sticks are needed to make a kite with 17 layers. 5814

49. Multi-Step  
Examine the pattern made by toothpick squares with increasing side lengths.

a. Write a sequence for the number of toothpicks added to form each new square. $a_n = 4n$

b. Write and evaluate a series in summation notation to represent the total number of toothpicks in a square with a side length of 6 toothpicks.

50. Critical Thinking  
Are the sums of $1 + 3 + 5 + 7 + 9$ and $9 + 7 + 5 + 3 + 1$ the same? Do these series have the same summation notation? Explain.

51. Write About It  
Explain why $S_n$ represents a partial sum and not a complete sum of the terms of a sequence.

52. Which notation accurately reflects the series $\sum_{k=1}^{7} (-1)^k (3k)$?

53. Which notation accurately reflects the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}$?

54. What is the value of $\sum_{k=1}^{3} k^2$?

55. Find the sum of the series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24}$.

876  Chapter 12  Sequences and Series
Find the first 5 terms of each sequence. (Lesson 12-1)

66. \(a_n = \left(\frac{1}{2}a + 2\right)^n\)

67. \(a_1 = 2, a_n = (a_{n-1})^2 - 1\)

68. \(a_n = \frac{e^n}{n}\)

25. 9, 49, 16, 81, 4

2, 3, 8, 63, 3968

2, 8, 32, 128, 512

Critical Thinking

What might the sum of the sequence \(1 - 1 + 1 - 1 + 1 - 1 + \cdots\) be if it continues forever? Explain.

Power Presentations

Write each series in summation notation.

1. \(1 - 10 + 100 - 1000 + 10,000\)

2. \(5 + 1 + 15 + 20 + 25\)

3. \(\sum_{k=1}^{n} (-1)^{k-1}\)

4. \(\sum_{k=1}^{n} \frac{5}{k}\)

5. \(\sum_{k=1}^{n} \frac{1}{5}\)

6. \(\sum_{k=1}^{n} \frac{k^2}{25}\)

7. Ann is making a display of hand-held computer games. There will be 1 game on top. Each row will have 8 additional games. She wants the display to have as many rows as possible with 100 games. How many rows will Ann’s display have?  

5

Also available on transparency

In Exercise 48c, some students might have difficulty recognizing how the properties correspond to the series \(\sum_{k=1}^{n} (3k^2 + 3k)\). Point out that \(a_k = 3k^2\) and \(b_k = 3k\), so \(\sum_{k=1}^{n} (3k^2 + 3k) = \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k\). Now students can use the formulas to evaluate each part for \(n = 17\) in Exercise 48c.

Journal

Have students explain summation notation using words and by providing an example. Encourage them to include an example of a partial sum as well.

Alternative Assessment

Have students create two different series using summation notation. One should be a series with alternating signs. Have students expand and evaluate each series for several different values of \(k\) and \(n\).
Evaluate Sequences and Series

Graphing calculators have built-in features that help you generate the terms of a sequence and find the sums of series.

Use with Lesson 12-2

Activity

Use a graphing calculator to find the first 7 terms of the sequence \(a_n = 1.5n + 4\). Then find the sum of those terms.

1. Find the first 7 terms of the sequence.

   Enter the LIST operations menu by pressing \(2\text{nd}\) \(STAT\) and scrolling right to the \(OPS\) menu. Then select the sequence command \(5\text{:seq}()\).

   The sequence command takes the following four expressions separated by commas.

   Enter the rule, using \(x\) as the variable. Enter 1 for the starting term and 7 for the ending term.

   Close the parentheses, and then press \(ENTER\).

   The terms of the sequence will be displayed in brackets. Use the arrow keys to scroll to see the rest of the terms.

   The first 7 terms are 5.5, 7, 8.5, 10, 11.5, 13, and 14.5.

2. Find the sum of the terms.

   Enter the LIST math menu by pressing \(2\text{nd}\) \(STAT\) and scrolling right to the \(MATH\) menu. Then select the sum command \(5\text{:sum}()\).

   Follow the steps for entering a sequence as shown in Step 1.

   The sum of the first 7 terms is 70.

Try This

Find the first 8 terms of each sequence. Then find the sum of those 8 terms.

1. \(a_n = 2n^2 - 5\)
2. \(a_n = \frac{1}{4}(2)^{n-1}\)
3. \(a_n = 0.3n + 1.6\)
4. \(a_n = 20n\)
5. \(a_n = n^2 - 2n\)
6. \(a_n = 0.1(5)^n\)

7. Critical Thinking Find the next 5 terms of the sequence 200, 182, 164, 146, 128, ... Then find the sum of those 5 terms.

Answers to Try This

1. \(-3, 3, 13, 27, 45, 67, 93, 123; 368\)
2. \(0.25, 0.5, 1, 2, 4, 8, 16, 32; 63.75\)
3. \(1.9, 2.2, 2.5, 2.8, 3.1, 3.4, 3.7, 4; 23.6\)
4. \(20, 40, 60, 80, 100, 120, 140, 160; 720\)
5. \(-1, 4, 21, 56, 115, 204, 329, 496; 1224\)
6. \(0.5, 2.5, 12.5, 62.5, 312.5, 1562.5, 7812.5, 39062.5; 48828\)
7. \(110, 92, 74, 56, 38; 370\)
12-3 Arithmetic Sequences and Series

Objectives
Find the indicated terms of an arithmetic sequence.
Find the sums of arithmetic series.

Vocabulary
arithmetic sequence
arithmetic series

California Standards
23.0 Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series. Also covered: 22.0

Who uses this?
You can use arithmetic sequences to predict the cost of mailing letters.

The cost of mailing a letter in 2005 gives the sequence 0.37, 0.60, 0.83, 1.06, ...
The sequence is called an arithmetic sequence because its successive terms differ by the same number \( d \) \((d \neq 0)\), called the common difference. For the mail costs, \( d \) is 0.23, as shown.

<table>
<thead>
<tr>
<th>Term</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.37</td>
<td>0.60</td>
<td>0.83</td>
<td>1.06</td>
</tr>
</tbody>
</table>

Differences 0.23 0.23 0.23

Recall that linear functions have a constant first difference. Notice also that when you graph the ordered pairs \((n, a_n)\) of an arithmetic sequence, the points lie on a straight line. Thus, you can think of an arithmetic sequence as a linear function with sequential natural numbers as the domain.

EXAMPLE 1
Identifying Arithmetic Sequences

Determine whether each sequence could be arithmetic. If so, find the common first difference and the next term.

A
\(-3, 2, 7, 12, 17, …\)

Differences
\(-3, 2, 7, 12, 17\)

The sequence could be arithmetic with a common difference of 5.
The next term is \(17 + 5 = 22\).

B
\(-4, -12, -24, -40, -60, …\)

Differences
\(-4, -12, -24, -40, -60\)

The sequence is not arithmetic because the first differences are not constant.

CHECK IT OUT

Determine whether each sequence could be arithmetic. If so, find the common difference and the next term.

1a. \(1, 2, 3, 4, 5, …\) arithmetic; \(-0.2, -0.2, -0.2, …\) not arithmetic

Each term in an arithmetic sequence is the sum of the previous term and the common difference. This gives the recursive rule \(a_n = a_{n-1} + d\). You also can develop an explicit rule for an arithmetic sequence.

Motivate
Pose this famous question to your students:
“What is the sum of the first 100 natural numbers?”

Explanations and answers are provided in Alternate Openers: Explorations Transparencies.

Power Presentations with PowerPoint
Find the 9th term of each sequence.
1. \(a_n = n + 6\) 11
2. \(a_n = 4 - n\) -1
3. \(a_n = 3n + 4\) 19

Write a possible explicit rule for the \(n\)th term of each sequence.
4. 4, 5, 6, 7, 8, … \(a_n = n + 3\)
5. -3, -1, 1, 3, 5, … \(a_n = 2n - 5\)
6. \(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots\) \(a_n = \frac{n}{2} + 1\)

Also available on transparency

Math Humor
Teacher: You have the terms of the sequence all mixed up.
Student: If you wanted them in order, why didn’t you just say so?

1 2 3 Organize
Objectives: Find the indicated terms of an arithmetic sequence.
Find the sums of arithmetic series.

Online Edition
Tutorial Videos, Interactivity, TechKeys
Chapter 12 Sequences and Series

Notice the pattern in the table. Each term is the sum of the first term and a multiple of the common difference.

This pattern can be generalized into a rule for all arithmetic sequences.

The \( n \)th term \( a_n \) of an arithmetic sequence is given by

\[
a_n = a_1 + (n - 1)d
\]

where \( a_1 \) is the first term and \( d \) is the common difference.

General Rule for Arithmetic Sequences

**Example 1**

Determine whether each sequence could be arithmetic. If so, find the common first difference and the next term.

A. \(-10, -4, 2, 8, 14, \ldots\)

yes; 6; 20

B. \(-2, -5, -11, -20, -32, \ldots\)

no

**Example 2**

Find the 12th term of the arithmetic sequence

\[20, 14, 8, 2, -4, \ldots\]

\(-46\)

Also available on transparency

**INTERVENTION**

Questioning Strategies

**EXAMPLE 1**

• Why do you have to find the first differences?
• How could you use the graph of a sequence to decide whether the sequence is arithmetic?

**EXAMPLE 2**

• What must you know to be able to find the \( n \)th term of a sequence? Explain.

**Technology**

To display the terms of an arithmetic sequence on a graphing calculator, enter the first term, press \( \boxed{+} \), enter the common difference, and press \( \boxed{\text{ENTER}} \) repeatedly.

**Finding the \( n \)th Term**

I like to check the value of a term by using a graphing calculator.

I enter the function for the \( n \)th, or general, term. For Example 2A, enter \( y = 32 + (x - 1)(-7) \).

I then use the table feature. Start at 1 (for \( n = 1 \)), and use a step of 1. Then find the desired term (\( y \)-value) as shown for \( n = 10 \).

**Student to Student**

Diana Watson

Bowie High School

**Guided Instruction**

Introduce an arithmetic sequence such as

\[4, 7, 10, 13, \ldots\]

... Make the connection with linear functions that students have already studied. Show students how the general rule for an arithmetic sequence can be used in a number of ways as you review Examples 2, 3, and 4. When introducing the sum formula for arithmetic series, you may want to remind students of the sum of a linear series (Lesson 12-2).

**Universal Access**

Through Modeling

Separate students into small groups and give each group a set of small manipulatives, such as centimeter squares or cubes (MK). Have students create patterns by adding a specified number of items to each row. As they construct the model, have them write down the number of items in each row and the cumulative total for each row. Then have them write a rule for the sequence that describes their pattern.


**Example 3**

Finding Missing Terms

Find the missing terms in the arithmetic sequence 11, 195, 120, -17.

Step 1 Find the common difference.

\[
\begin{align*}
    a_n &= a_1 + (n - 1)d \\
    a_{14} &= a_9 + (14 - 9)d \\
    195 &= a_9 + 5d \\
    120 &= a_9 + 120 \\
    75 &= 5d \\
    15 &= d
\end{align*}
\]

Find 9th term using 15 for d.

\[
\begin{align*}
    a_9 &= 120 + 5d \\
    &= 120 + 5(15) \\
    &= 120 + 75 \\
    &= 195
\end{align*}
\]

\[
\begin{align*}
    a_{14} &= a_9 + 14d \\
    &= 195 + 75d \\
    &= 195 + 75(15) \\
    &= 195 + 1125 \\
    &= 1320
\end{align*}
\]

3. Find the missing terms in the arithmetic sequence 
\[
2, 0, 3, \frac{3}{2}, 1, \frac{1}{2}
\]

Because arithmetic sequences have a common difference, you can use any two terms to find the difference.

**Example 4**

Finding the nth Term Given Two Terms

Find the 6th term of the arithmetic sequence with \(a_8 = 120\) and \(a_{14} = 195\).

Step 1 Find the common difference.

\[
\begin{align*}
    a_n &= a_1 + (n - 1)d \\
    a_{14} &= a_8 + 14d \\
    195 &= 120 + 120d \\
    15 &= 120d \\
    d &= \frac{15}{120} \\
    &= \frac{1}{8}
\end{align*}
\]

Step 2 Find \(a_6\).

\[
\begin{align*}
    a_n &= a_1 + (n - 1)d \\
    a_6 &= a_1 + 5d \\
    &= a_1 + 5 \left( \frac{1}{8} \right) \\
    &= a_1 + \frac{5}{8}
\end{align*}
\]

Step 3 Write a rule for the sequence, and evaluate to find \(a_6\).

\[
\begin{align*}
    a_n &= a_1 + (n - 1)d \\
    a_6 &= a_1 + (6 - 1)d \\
    &= a_1 + 5 \left( \frac{1}{8} \right) \\
    &= a_1 + \frac{5}{8}
\end{align*}
\]

Find the 5th term of the arithmetic sequence.

\[
\begin{align*}
    a_5 &= a_1 + (5 - 1)d \\
    &= a_1 + 4 \left( \frac{1}{8} \right) \\
    &= a_1 + \frac{4}{8} \\
    &= a_1 + \frac{1}{2}
\end{align*}
\]

Find the missing terms using \(a_1\) or \(a_5\) and \(a_8\) or \(a_{14}\). Discuss how \(n\) is different from \(a_n\).

**Power Presentations with PowerPoint**

**Additional Examples**

**Example 3**

Find the missing terms in the arithmetic sequence 
\(17, 6, 5, \ldots, -7\), \(11, 5, -1\)

**Example 4**

Find the 5th term of the arithmetic sequence with \(a_6 = 85\) and \(a_{14} = 157\). 49

**INTERVENTION**

**Questioning Strategies**

**Example 3**

- What is the first step in finding the common difference? Explain.
- After you find the common difference, is there a way to find the missing terms other than using the rule for the nth term? Explain.

**Example 4**

- Why can you replace \(a_6\) and \(a_1\) with other terms from the sequence?

**Visual** It often helps to visually represent the sequences with missing terms. For **Example 4**, you may want to show something like this.

<table>
<thead>
<tr>
<th>(a_9)</th>
<th>(a_{10})</th>
<th>(a_{11})</th>
<th>(a_{12})</th>
<th>(a_{13})</th>
<th>(a_{14})</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
</tr>
<tr>
<td>195</td>
<td>195</td>
<td>195</td>
<td>195</td>
<td>195</td>
<td>195</td>
</tr>
</tbody>
</table>

**Teacher to Teacher**

I tell students that \(a_n = a_1 + (n - 1)d\) works with any two terms. For **Example 4**:

\[
\begin{align*}
    a_{14} &= a_9 + (14 - 9)d \\
    195 &= 120 + 5d \\
    75 &= 5d \\
    15 &= d
\end{align*}
\]

Then solve for \(a_6\) by using \(a_9 = a_6 + (9 - 6)d\), where \(a_9 = 120\) and \(d = 15\). In a few steps, you find that \(a_6 = 75\). You don't ever need to find \(a_1\). Students really like this and it always works.

Dave Barker
Los Alamitos, CA
In Lesson 12-2 you wrote and evaluated series. An arithmetic series is the indicated sum of the terms of an arithmetic sequence. You can derive a general formula for the sum of an arithmetic series by writing the series in forward and reverse order and adding the results.

\[ S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n \]
\[ S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1 \]

\[ 2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) \cdots + (a_1 + a_n) \]
\[ (a_1 + a_n) \text{ is added } n \text{ times} \]

\[ 2S_n = n(a_1 + a_n) \]
\[ S_n = \frac{n(a_1 + a_n)}{2}, \text{ or } S_n = n\left(\frac{a_1 + a_n}{2}\right) \]

### Example 5

Find the indicated sum for each arithmetic series.

A. \( S_{18} \) for \( 13 + 2 + (-9) + (-20) + \cdots \)

B. \( \sum_{k=1}^{15} (5 + 2k) \)

### Example 6

The center section of a concert hall has 15 seats in the first row and 2 additional seats in each subsequent row.

A. How many seats are in the 20th row? Answer: 53

B. How many seats in total are in the first 20 rows? Answer: 680

### Additional Examples

- **Example 5**
  - Find the indicated sum for each arithmetic series.
  - **A.** \( S_{18} \) for \( 13 + 2 + (-9) + (-20) + \cdots \)
  - **B.** \( \sum_{k=1}^{15} (5 + 2k) \)

- **Example 6**
  - The center section of a concert hall has 15 seats in the first row and 2 additional seats in each subsequent row.
  - **A.** How many seats are in the 20th row? Answer: 53
  - **B.** How many seats in total are in the first 20 rows? Answer: 680

### Questioning Strategies

**Example 5**

- How do you know which term(s) to find to evaluate the sum?
- When do you find the common difference to find the sum? Explain.

**Example 6**

- Why do you use an arithmetic sequence to solve the problem?
- How do you know which part of the problem is solved using an arithmetic sequence and which is solved using an arithmetic series?
**Theater Application**

The number of seats in the first 14 rows of the center orchestra aisle of the Marquis Theater on Broadway in New York City form an arithmetic sequence as shown.

A How many seats are in the 14th row?

Write a general rule using \(a_1 = 11\) and \(d = 1\).

\[
a_n = a_1 + (n - 1)d \quad \text{Explicit rule for } n\text{th term}
\]

\[
a_{14} = 11 + (14 - 1)1 \quad \text{Substitute.}
\]

\[
= 11 + 13
\]

\[
= 24 \quad \text{Simplify.}
\]

There are 24 seats in the 14th row.

B How many seats in total are in the first 14 rows?

Find \(S_{14}\) using the formula for finding the sum of the first \(n\) terms.

\[
S_n = \frac{n(a_1 + a_n)}{2} \quad \text{Formula for first } n\text{ terms}
\]

\[
S_{14} = 14 \left(\frac{11 + 24}{2}\right) \quad \text{Substitute.}
\]

\[
= 14 \left(\frac{35}{2}\right)
\]

\[
= 245 \quad \text{Simplify.}
\]

There are 245 seats in rows 1 through 14.

6. **What if…?** Suppose that each row after the first had 2 additional seats.

a. How many seats would be in the 14th row? **37 seats**

b. How many total seats would there be in the first 14 rows? **336 total seats**

**THINK AND DISCUSS**

1. Compare an arithmetic sequence with a linear function.

2. Describe the effect that a negative common difference has on an arithmetic sequence.

3. Explain how to find the 6th term in a sequence when you know the 3rd and 4th terms.

4. Explain how to find the common difference when you know the 7th and 12th terms of an arithmetic sequence.

5. **GET ORGANIZED** Copy and complete the graphic organizer. Write in each rectangle to summarize your understanding of arithmetic sequences.

**Summary**

Ask students how to identify an arithmetic sequence. Then have them explain how to use the general rule for arithmetic sequences to find the \(n\)th term. An arithmetic sequence has a common first difference; to find the \(n\)th term, use \(a_n = a_1 + (n - 1)d\), where \(a_1\) is the first term and \(d\) is the common difference.

**Answers to Think and Discuss**

Possible answers:

1. An arithmetic sequence is a linear function with the domain restricted to the natural numbers.

2. decreases with each new term

3. Find \(d\) by subtracting \(a_2 - a_1\), use the general rule to solve for \(a_1\), and then evaluate for \(n = 6\).

4. Substitute \(a_1\) for \(a_1\) and \(a_{12}\) for \(a_n\) in the general rule and solve for \(d\).

GUIDED PRACTICE

1. Vocabulary The expression 10 + 20 + 30 + 40 + 50 is an arithmetic sequence.

2. Determine whether each sequence could be arithmetic. If so, find the common difference and the next term.
   a. 46, 39, 32, 25, 18, ...
   b. 28, 21, 15, 10.6, ...
   c. 12, 10, 8, 6, 4, ...
   d. 3, 8, 13, 18, ...
   e. 6, 9, 12, 15, ...
   f. 7, −3.2, −3.4, −3.6, −3.8, ...

3. Find the 8th term of each arithmetic sequence.
   a. 38
   b. 10
   c. 1.4

4. Find the missing terms in each arithmetic sequence.
   a. 2.5
   b. 37
   c. −1

5. Find the 9th term of each arithmetic sequence.
   a. 11
   b. 13
   c. 16

6. Find the indicated sum for each arithmetic series.
   a. 444
   b. 117

7. Salary Juan has taken a job with an initial salary of $26,000 and annual raises of $1250.
   a. What will his salary be in his 6th year? $32,250
   b. How much money in total will Juan have earned after six years? $174,750

PRACTICE AND PROBLEM SOLVING

Determine whether each sequence could be arithmetic. If so, find the common difference and the next term.

21. 288, 144, 72, 36, 18, ...
22. −2, −12, −22, −32, −42, ...
23. 0.99, 0.9, 0.81, 0.72, ...
24. 12, 11.9, 11.8, 11.7, ...
25. 2, 3, 4, 12, ...
26. −3, −2.5, −2, −1.5, ...
27. 2.3, 3.2, 4.1, ...
28. 66, 65, 44, ...
29. 1, 1.9, 1.7
30. 2.3, 4.1, 6, ...
31. 0.2, 0.4, 0.6, ...
32. 15
33. $5 for 884 Chapter 12 Sequences and Series
36. **Consumer Economics** Clarissa is buying a prom dress on layaway. She agrees to make a $15 payment and increase the payment by $5 each week.
   a. What will her payment be in the 9th week? $55
   b. How much money in total will Clarissa have paid after 9 weeks? $315

37. **Clocks** A clock chimes every hour. The clock chimes once at 1 o’clock, twice at 2 o’clock, and so on.
   a. How many times will the clock chime from 1 P.M. through midnight? In exactly 24-hour period?
   b. **What If...?** Another clock also chimes once on every half hour. How does this affect the sequence and the total number of chimes per day?

Find the indicated sum for each arithmetic series.

38. \[ \sum_{k=1}^{10} (555 - 11k) \]
39. \[ \sum_{k=1}^{15} (4 - 0.5k) \]
40. \[ \sum_{k=1}^{18} (-33 + \frac{5}{2}k) \]

41. \( S_{16} \) for 7.5 + 7 + 6.5 + 6.0 + \ldots
42. \( S_{18} \) for 2 + 9 + 16 + 23 + \ldots

43. **Architecture** The Louvre pyramid in Paris, France, is built of glass panes. There are 4 panes in the top row, and each additional row has 4 more panes than the previous row.
   a. Write a series in summation notation to describe the total number of glass panes in \( n \) rows of the pyramid.
   b. If the pyramid were made of 18 complete rows, how many panes would it have?
   c. The actual pyramid has 11 panes less than a complete 18-row pyramid because of the space for the entrance. Find the total number of panes in the Louvre pyramid.

44. **Physics** Water towers are tall to provide enough water pressure to supply all of the houses and businesses in the area of the tower. Each foot of height provides 0.43 psi (pounds per square inch) of pressure.
   a. Write a sequence for the pressure in psi for each foot of height.
   b. What is the minimum height that supplies 50 psi, a typical minimum supply pressure? **about 116.3 ft**
   c. What is the minimum height that supplies 100 psi, which is a typical maximum pressure? **about 232.6 ft**
   d. Graph the sequence, and discuss the relationship between the height found for a pressure of 50 psi and the height found at 100 psi.

45. **Exercise** Sheila begins an exercise routine for 20 minutes each day. Each week she plans to add 5 minutes per day to the length of her routine.
   a. For how many minutes will she exercise each day of the 6th week? **45 minutes**
   b. What happens to the length of Sheila’s exercise routine if she continues this increasing pattern for 2 years?

46. **Geology** Every year the continent of North America moves farther away from Europe.
   a. How much farther from Europe will North America be in 50 years?
   b. How many years until an extra mile is added? (Hint: 1 mi = 1609 m) **about 69,956 yr**

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12-3 Arithmetic Sequences and Series

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**Answers**

37a. 78; 156
   b. adds 1 to each term of the sequence; adds 24 to the total number per day

44d. The height is twice as great because the graph represents a direct variation function.

---

**Practice C**

Find the 21st term of each arithmetic sequence.
1. \( 1, 3, 5, 7, 9, \ldots \)
2. \( 8, 10, 12, 14, \ldots \)
3. \( 30, 26, 22, 18, \ldots \)

Find the missing terms in each arithmetic sequence.
4. \( 3, 5, 7, 9, \ldots \)
5. \( 10, 15, 20, 25, \ldots \)
6. \( 10, 15, 20, 25, \ldots \)

Find the 12th term of each arithmetic sequence.
7. \( a_1 = 10, d = 4 \)
8. \( a_1 = 2, d = -4 \)
9. \( a_1 = -3, d = 5 \)

Find the indicated sum for each arithmetic series.
10. \( S_{10} \) for \( 1 + 3 + 5 + 7 + \ldots \)
11. \( S_{15} \) for \( 2 + 4 + 6 + 8 + \ldots \)
12. \( S_{20} \) for \( 3 + 6 + 9 + 12 + \ldots \)

---

**Note:** A family celebrates each birthday of quadruplet children with individual cake Decorating. The last year such a pattern occurred and the number of candles on the cake represented the quadruplet’s age. Four candles are used every year. How many candles are used in 21 years? **1155 candles**
Exercise 47 involves using an arithmetic series to identify the number of tetrahedrons needed to make a kite. This exercise prepares students for the Concept Connection on page 888.

In Exercise 52, students who choose A, C, or D may not recognize that arithmetic sequences must have a common difference. Remind students that they can check first differences to determine whether any sequence could be arithmetic.

Answers

50. Possible answer: 2 terms; yes, an arithmetic sequence is a linear function with a limited domain.

51. Possible answer: If \( d > 0 \), the value of the \( n \)th term increases toward infinity. If \( d < 0 \), the value of the \( n \)th term decreases toward negative infinity.

58a. \[ a_n = a_1 + (n - 1)d \]
\[ (a_m - a_n) = (n - 1)d - (m - 1)d \]
\[ a_n - a_m = (n - 1)d - (m - 1)d \]
\[ d = \frac{a_n - a_m}{n - m} \]

59. \[ S_n = \frac{2a_1 + (n - 1)d}{2} \]

when you do not know the last term but know 2 of the 3 remaining variable quantities, \( S_n, \) \( n, \) or \( a_1 \).

61. The equality states that the value of a term, such as \( a_6 \), is twice the value of the term that has half its term number, such as \( a_3 \). This gives \( a_6 = 2a_3 \). For this to be true, the first term must equal the common difference, or \( a_1 = d \).

49a. 6th, 11th, 16th, 21st, and 26th Streets

49. Sports A town is planning a 5K race. The race route will begin at 1st street, travel 30 blocks down Main Street, and finish on 31st Street. The race planners want to have water stations at each turn. In addition, they want to place 5 more water stations evenly distributed between 1st Street and 31st Street on Main Street.

a. At what street intersections should the water stations be placed?

b. If each block is 0.1 mile, what is the maximum distance a runner will be from a water station while on Main Street? 0.25 mi

50. Critical Thinking What is the least number of terms you need to write the general rule for an arithmetic sequence? How many points do you need to write an equation of a line? Are these answers related? Explain.

51. Write About It An arithmetic sequence has a positive common difference. What happens to the \( n \)th term as \( n \) becomes greater and greater? What happens if the sequence has a negative common difference?

52. Which sequence could be an arithmetic sequence?

- \[ 2, 3, 4, 5, \ldots \]
- \[ 1, 1, 1, 1 \ldots \]
- \[ 2, 4, 6, 8, \ldots \]
- \[ 45, 57, 69, 81, 93, \ldots \]

53. A catering company charges a setup fee of $45 plus $12 per person. Which of the following sequences accurately reflects this situation?

- \[ a_n = 45 + 12(n - 1) \]
- \[ a_n = 57 + 12n \]
- \[ a_n = 79 + 12(n - 1) \]

54. Finance The starting salary for a summer camp counselor is $395 per week. In each of the subsequent weeks, the salary increases by $45 to encourage experienced counselors to work for the entire summer. If the salary is $710 in the last week of the camp, for how many weeks does the camp run? 8 weeks

886 Chapter 12 Sequences and Series
54. Which graph might represent the terms of an arithmetic sequence?

55. Given the arithmetic sequence 4, 11, 18, 25, what are the three missing terms?

56. Which represents the sum of the arithmetic sequence 19, 16, 13, 10 + 7 + 4?

57. Gridded Response What is the 13th term of the arithmetic sequence

58. Consider the two terms of an arithmetic series \( a_n \) and \( a_{n+1} \).
   a. Show that the common difference is \( d = \frac{a_{n+1} - a_n}{n - n+1} \).
   b. Use the new formula to find the common difference for the arithmetic sequence with \( a_2 = 88 \) and \( a_3 = 304 \).

59. Find a formula for the sum of an arithmetic sequence that does not include the last term. When might this formula be useful?

60. The sum of three consecutive terms of an arithmetic sequence is 60. If the product of these terms is 7500, what are the terms?

61. Critical Thinking What does \( a_{2n} = 2a_n \) mean and for what arithmetic sequences is it true?

### CHALLENGE AND EXTEND

62. Decay \( f(x) = 1.25(0.75)^x \)

63. Growth \( f(x) = 1.43(3.52)^x \)

64. Decaying \( f(x) = 0.92(0.64)^x \)

65. Sound The loudness of sound is given by \( L = 10 \log \left( \frac{I}{I_0} \right) \), where \( L \) is the loudness of sound in decibels, \( I \) is the intensity of sound, and \( I_0 \) is the intensity of the softest audible sound. A sound meter at an auto race had a relative intensity of 10^7.2 dB. Find the loudness of the sound in decibels. (Lesson 7-3) 92 dB

66. Write each series in summation notation. (Lesson 12-2)

67. \( \sum_{k=1}^{12} (k^2 - 2) \)

68. \( \sum_{k=1}^{14} (k - 1)^2 \)

### 12-3 PROBLEM SOLVING

#### 12-3 Challenge

A busy tour guide introduces mathematics to her tour group. She explains that the tour numbers from 1 to 100 are seen on the buildings of the city. To see the diversity of views, the tour guide points out buildings that are 

1. Prime numbers
2. Perfect squares
3. Positive multiples of 4
4. Positive multiples of 6

Choose the letter for the best answer.

Choose the letter for the best answer.

#### 12-3 Lesson Quiz

1. Determine whether the sequence could be arithmetic. If so, find the first difference and the next term.
   \(-1, -4, -7, -10, -13, \ldots \)
   yes; \(-3, -16\)

2. Find the 10th term of the arithmetic sequence
   \(-2, -5, -8, -11, -14, \ldots, -29\)

3. Find the missing terms in the arithmetic sequence
   \(15, 30, 45, \ldots, 17\)

4. Find the 6th term of the arithmetic sequence with \(a_1 = 64\) and \(a_{12} = 88\).

5. Find the indicated sum for \(\sum_{k=1}^{12} (15 - 4k)\).

6. The side section of an auditorium has 12 seats in the first row and 3 additional seats in each subsequent row. How many seats are in the 10th row? How many seats in total are in the first 10 rows?

### Math Background

An alternative formula for the sum of an arithmetic series is \(S_n = \frac{n}{2}(a_1 + a_n) + (n - 1)d\). Students are asked to find this formula in Exercise 59.

### Journal

Have students explain how to use the formula for the sum of the first \(n\) terms of an arithmetic sequence. Ask them to include an example.

### Alternative Assessment

Have students create an arithmetic sequence. Have them identify the common difference and several different terms of the sequence. Have them show how the general rule and the sum formula work with the sequence they have created.

### Power Presentations with PowerPoint

1. **Lesson 7-2 Quiz**
   - Determine whether the sequence could be arithmetic. If so, find the difference and the next term.
   - Yes; \(-3, -16\)

2. **Lesson 12-3 Quiz**
   - Find the 10th term of the arithmetic sequence
   - \(-2, -5, -8, -11, -14, \ldots, -29\)

3. **Lesson 12-3 Quiz**
   - Find the missing terms in the arithmetic sequence
   - \(15, 30, 45, \ldots, 17\)

4. **Lesson 12-3 Quiz**
   - Find the 6th term of the arithmetic sequence with \(a_1 = 64\) and \(a_{12} = 88\).

5. **Lesson 12-3 Quiz**
   - Find the indicated sum for \(\sum_{k=1}^{12} (15 - 4k)\).

6. **Lesson 12-3 Quiz**
   - The side section of an auditorium has 12 seats in the first row and 3 additional seats in each subsequent row. How many seats are in the 10th row? How many seats in total are in the first 10 rows?

Also available on transparent text.
Exploring Arithmetic Sequences and Series

Go Fly a Kite! Alexander Graham Bell, the inventor of the telephone, is also known for his work with tetrahedral kites. In 1902, Bell used the kites to prove that it is possible to build an arbitrarily large structure that will fly. The kites are made up of tetrahedrons (four-sided triangular figures) with two sides covered with fabric. As shown in the figure, the size of a tetrahedral kite is determined by how many layers it has.

1. The first layer of a tetrahedral kite has 1 tetrahedron, the second layer has 3 tetrahedrons, and the third layer has 6 tetrahedrons. Write a sequence that shows how many tetrahedrons are in each of the first 10 layers.

2. Write a recursive formula for the sequence.

3. Write an explicit rule for the $n$th term of the sequence.

4. How many tetrahedrons are there in the 25th layer?

5. Write a series in summation notation that gives the total number of tetrahedrons in a kite with 25 layers.

6. Evaluate the series in problem 5 to find the total number of tetrahedrons in a kite with 25 layers.

7. You see someone flying a large tetrahedral kite at a kite festival. You look up and estimate that the bottom layer of the kite contains between 100 and 110 tetrahedrons. How many layers does the kite have? How many tetrahedrons did it take to build the kite?

INTERVENTION

Scaffolding Questions

1. What pattern do you notice in the first few terms of the sequence? The first differences are 2, 3, and 4.

2. How can you find the 5th term when you know the 4th term? Add 5 to the 4th term.

3. What do the differences tell you about the sequence? It is quadratic.


5. How would you write the first few terms and the last term of this series in expanded notation? $1 + 3 + 6 + \ldots + 325$

6-7. How can you apply summation formulas to find the sum? Rewrite the series as the sum of a linear and a quadratic series.

Extension

Is it possible to make a tetrahedral kite in which the bottom layer has exactly 360 tetrahedrons? no
Quiz for Lessons 12-1 Through 12-3

12-1 Introduction to Sequences
Find the first 5 terms of each sequence.
1. \(a_n = \frac{2n}{3}\)
2. \(a_n = 4^n - 1\)
3. \(a_1 = -1\) and \(a_n = 2a_{n-1} - 12\)
4. \(a_n = n^2 - 2n - \frac{1}{3}\)

Write a possible explicit rule for the nth term of each sequence.
5. \(a_n = 3n + 5\)
6. \(a_n = -2n^2\)
7. \(1000, 200, 40, 8, \ldots\)

Expand each series and evaluate.
10. \(-16 + (-18) + (-20) + (-22) = -76\)
11. \(\sum_{k=1}^{4} \frac{1}{k+2} = \frac{5}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{21}{10}\)

12-2 Series and Summation Notation
Expand each series and evaluate.
12. \(\sum_{k=1}^{4} (-1)^k (k^2 - 2) = 1 + 2 - 7 + 14 - 23 = -13\)
Evaluate each series.
13. \(\sum_{k=1}^{5} \frac{k}{2} = \frac{15}{2}\)
14. \(\sum_{k=1}^{5} 22k = 22140\)
15. \(\sum_{k=1}^{5} k = 120\)

16. The first row of a theater has 20 seats, and each of the following rows has 3 more seats than the preceding row. How many seats are in the first 12 rows? 438

12-3 Arithmetic Sequences and Series
Find the nth term of each arithmetic sequence.
17. \(10.00, 10.11, 10.22, 10.33, \ldots, 10.77\)
18. \(-5, -13, -21, -29, \ldots, -61\)
19. \(a_2 = 57.5\) and \(a_3 = 80\)
20. \(a_{10} = 141\) and \(a_{13} = 186\)

Find the missing terms in each arithmetic sequence.
21. \(-23, \ldots, -89, -45, -67, \ldots, -79\)
22. \(31, \ldots, 79, 43, 55, 67, \ldots\)

Find the indicated sum for each arithmetic series.
23. \(S_{11}\) for \(40 + 30 + 20 + 10 + \cdots + 10 = 104\)
24. \(\sum_{k=5}^{11} 4k = 93.5\)
25. \(\sum_{k=1}^{11} (0.5k + 5.5) = 371\)
26. \(S_{14}\) for \(-6 - 1 + 4 + 9 + \cdots\)

27. Suppose that you make a bank deposit of $1 the first week, $1.50 the second week, $2 the third week, and so on. How much will you contribute to the account on the last week of the year (52nd week)? What is the total amount that you have deposited in the bank after one year? $28.50; $715

Answers.
9. For graph, see p. A46.
# Exploring Geometric Sequences and Series

## One-Minute Section Planner

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Lab Resources</th>
<th>Materials</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Lesson 12-4 Geometric Sequences and Series</strong>&lt;br&gt;• Find terms of a geometric sequence, including geometric means.&lt;br&gt;• Find the sums of geometric series.&lt;br&gt;☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td><strong>Technology Lab Activities</strong>&lt;br&gt;12-4 Technology Lab</td>
<td><strong>Optional</strong>&lt;br&gt;graphing calculator</td>
</tr>
<tr>
<td><strong>12-5 Algebra Lab Explore Infinite Geometric Series</strong>&lt;br&gt;• Use a sequence of squares to explore an infinite geometric series.&lt;br&gt;☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td><strong>Algebra Lab Activities</strong>&lt;br&gt;12-5 Lab Recording Sheet</td>
<td><strong>Required</strong>&lt;br&gt;graph paper, graphing calculator</td>
</tr>
<tr>
<td><strong>Lesson 12-5 Mathematical Induction and Infinite Geometric Series</strong>&lt;br&gt;• Find sums of infinite geometric series.&lt;br&gt;• Use mathematical induction to prove statements.&lt;br&gt;☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td></td>
<td><strong>Optional</strong>&lt;br&gt;dominoes or books, graphing calculator</td>
</tr>
<tr>
<td><strong>Extension Area Under a Curve</strong>&lt;br&gt;• Approximate area under a curve by using rectangles.&lt;br&gt;☐ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests</td>
<td></td>
<td><strong>Optional</strong>&lt;br&gt;graphing calculator</td>
</tr>
</tbody>
</table>

MK = Manipulatives Kit
Math Background:
Teaching the Standards

GEOMETRIC SEQUENCES
Lesson 12-4

In a geometric sequence, the ratio of successive terms is a constant called the common ratio \( r \) (\( r \neq \pm 1 \)). The \( n \)th term of a geometric sequence can be found by multiplying the first term by \( r \) \((n - 1) \) times; equivalently, \( a_n = a_1 r^{n-1} \). This general rule is analogous to the equation of an exponential function, \( y = ab^x \), and a geometric sequence may be viewed as an exponential function whose domain is restricted to the natural numbers.

For \( a_1 > 0 \), the value of \( r \) determines the nature of the geometric sequence.

- When \( r > 1 \), the sequence represents exponential growth. The terms of the sequence become arbitrarily large.
- When \( 0 < r < 1 \), the sequence represents exponential decay. The terms of the sequence become arbitrarily small. That is, they approach (but never equal) 0.
- When \( r < 0 \), the signs of the terms alternate between positive and negative.

Given two terms of a geometric sequence, \( a_m \) and \( a_n \), the relationship \( a_n = a_m r^{n-m} \) may be used to calculate the common ratio. Notice that this formula holds whether \( m < n \) or \( m > n \), and that the formula reduces to the general rule for the \( n \)th term when \( m = 1 \).

GEOMETRIC SERIES
Lessons 12-4, 12-5

A geometric series is a series whose terms form a geometric sequence. The partial sum \( S_n \) of the first \( n \) terms of a geometric series is given by \( S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} \). Multiplying each side of this equation by \( r \) gives \( rS_n = a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + a_1 r^n \). Subtracting \( rS_n \) from \( S_n \) leads to the general formula for the partial sum of the first \( n \) terms of a geometric series,

\[
S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right).
\]

It is also possible to find the sum \( S \) of an infinite geometric series when \( |r| < 1 \). In this case, the sum is defined to be the limit of the partial sums as \( n \) approaches infinity. That is, \( S = \lim_{n \to \infty} S_n \).

When \( |r| < 1 \), as \( n \) becomes larger and larger, the values of \( r^n \) approach 0. Therefore, when \( n \) is very large, \( r^n \) has a negligible effect on the value of \( S_n \). Loosely speaking, \( r^n \) “disappears” from the numerator as \( n \) approaches infinity, leaving \( S = \frac{a_1}{1 - r} \).

When \( |r| \geq 1 \), the partial sums \( S_n \) increase without bound and the sum of the infinite series is not defined. Students will see a more rigorous treatment of these ideas when they study limits in a calculus course.

MATHEMATICAL INDUCTION
Lessons 12-4, 12-5

Mathematical induction is a method of proof used to demonstrate that a statement is true for all natural numbers. The first known use of the technique is credited to an Italian mathematician, Francesco Maurolico, who used mathematical induction in 1575 to prove that the sum of the first \( n \) odd numbers is \( n^2 \). Students should be aware that although the name of the method includes the word induction, the method is an example of deductive reasoning rather than inductive reasoning.

To use mathematical induction to prove that a statement is true for all natural numbers \( n \), two things must be shown.

1. The statement is true for \( n = 1 \).
2. If the statement is true for any natural number \( k \), then the statement is true for \( k + 1 \).

The rationale for these two parts of the proof is analogous to the way in which a row of dominos may be toppled. The first step of a proof by mathematical induction corresponds to tipping over the first domino. The second step of the proof says that when one domino falls, it always knocks over the next one. Putting these two ideas together shows that the entire chain of dominos falls. In other words, the statement that is true for \( n = 1 \) is in fact true for all subsequent values of \( n \).
Objectives: Find terms of a geometric sequence, including geometric means.

Find the sums of geometric series.

**Vocabulary**
- geometric sequence
- geometric mean
- geometric series

**California Standards**
- 23.0 Students derive the summation formulas for geometric series and for both finite and infinite geometric series.

**Objectives**
Find terms of a geometric sequence, including geometric means.
Find the sums of geometric series.

Serena Williams was the winner out of 128 players who began the 2003 Wimbledon Ladies’ Singles Championship. After each match, the winner continues to the next round and the loser is eliminated from the tournament. This means that after each round only half of the players remain.

The number of players remaining after each round can be modeled by a geometric sequence. In a geometric sequence, the ratio of successive terms is a constant called the common ratio \( r \). For the players remaining, \( r = \frac{1}{2} \).

<table>
<thead>
<tr>
<th>Term</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>16</td>
</tr>
</tbody>
</table>

**Identifying Geometric Sequences**
Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.

**Solution**
- \( A \): geometric; \( r = \frac{1}{2} \)
- \( B \): arithmetic; \( d = -4 \)
- \( C \): neither

**Explorations and answers are provided in Alternate Openers: Explorations Transparencies.**
Each term in a geometric sequence is the product of the previous term and the common ratio, giving the recursive rule for a geometric sequence.

\[ a_n = a_{n-1} \times r \]

You can also use an explicit rule to find the \( n \)th term of a geometric sequence. Each term is the product of the first term and a power of the common ratio as shown in the table.

**General Rule for Geometric Sequences**

The \( n \)th term \( a_n \) of a geometric sequence is

\[ a_n = a_1 r^{n-1} \]

where \( a_1 \) is the first term and \( r \) is the common ratio.

**Example 2**

Finding the \( n \)th Term Given a Geometric Sequence

Find the 9th term of the geometric sequence \(-5, 10, -20, 40, -80, \ldots\).

Step 1 Find the common ratio.

\[ r = \frac{a_2}{a_1} = \frac{10}{-5} = -2 \]

Step 2 Write a rule, and evaluate for \( n = 9 \).

\[ a_n = a_1 r^{n-1} \quad \text{General rule} \]

\[ a_9 = -5(-2)^{9-1} \quad \text{Substitute } -5 \text{ for } a_1, 9 \text{ for } n, \text{ and } -2 \text{ for } r. \]

\[ = -5(256) = -1280 \]

The 9th term is \(-1280\).

**Check** Extend the sequence.

\[ a_2 = -80 \quad \text{Given} \]

\[ a_3 = -80(-2) = 160 \]

\[ a_4 = 160(-2) = -320 \]

\[ a_5 = -320(-2) = 640 \]

\[ a_6 = 640(-2) = -1280 \checkmark \]

Find the 9th term of each geometric sequence.

2a. \( \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \frac{3}{64} \ldots \)

2b. 0.001, 0.01, 0.1, 1, 10, 100, 1000, 10000

**Additional Examples**

**Example 1**

Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.

A. 100, 93, 86, 79, ... arithmetic; \( d = -7 \)

B. 180, 90, 60, 15, ... neither

C. 5, 1, 0.2, 0.04, ... geometric; \( r = 0.2 \)

**Example 2**

Find the 7th term of the geometric sequence 3, 12, 48, 192, ....

12,288

Also available on transparency

**INTERVENTION**

**Questioning Strategies**

**EXAMPLE 1**

- How would you determine whether a sequence is arithmetic or geometric?
- How could you try to find a possible rule for a sequence that is neither arithmetic nor geometric?

**EXAMPLE 2**

- How could you check the value you found for \( r \), the common ratio?
- How could you predict the sign of the 9th term before you calculate it?

**Technology**

To display the terms of a geometric sequence on a graphing calculator, enter the first term, press \( X \) \( \times \) \( \), enter the common ratio, and press \( \) \( \), repeatedly.

**2 Teach**

**Guided Instruction**

Review with students the domain and range for sequences with values in the form \((n, a_n)\). Help students recall that the domain, or set of values for \( n \), is the natural numbers, while the range is some set of real numbers.

**Inclusion** Make sure that students realize that \( r \) cannot be 0 and that the value of \( a_n \) cannot be 0.

**Universal Access**

**Through Graphic Organizers**

Have students make a table for geometric sequences, labeling the rows so that the first row shows the values of \( n \), the second row shows the value of \( a_n \), and the third row shows the ratio between terms. The table for **Example 2** is shown below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>-5</td>
<td>10</td>
<td>-20</td>
<td>-40</td>
<td>-80</td>
<td>-160</td>
<td>-320</td>
</tr>
<tr>
<td>( r )</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
</tr>
</tbody>
</table>
Chapter 12 Sequences and Series

Example 3
Finding the \( n \)th Term Given Two Terms

Find the 10th term of the geometric sequence with \( a_3 = 36 \) and \( a_1 = 324. \)

Step 1
Find the common ratio.

\[
\begin{align*}
\frac{a_7}{a_5} &= r^{7-5} \\
384 &= a_5 \cdot 96 \\
4 &= r^2 \quad \text{Simplify.}
\end{align*}
\]

Step 2
Find \( a_1. \)

Consider both the positive and negative values for \( r. \)

\[
\begin{align*}
a_n &= a_1 \cdot r^{n-1} \\
a_5 &= a_1 \cdot 2^{5-1} \quad \text{General rule}
\end{align*}
\]

\[
\begin{align*}
96 &= a_1 \cdot 4 \\
6 &= a_1 \quad \text{Divide both sides by 46.}
\end{align*}
\]

\[
\begin{align*}
\pm 2 &= r \quad \text{Take the square root of both sides.}
\end{align*}
\]

Step 3
Write the rule and evaluate for \( a_{10}. \)

Consider both the positive and negative values for \( r. \)

\[
\begin{align*}
a_n &= a_1 \cdot r^{n-1} \\
a_n &= a_1 \cdot r^{10-1} \quad \text{General rule}
\end{align*}
\]

\[
\begin{align*}
a_{10} &= a_1 \cdot 2^{10-1} \quad \text{Evaluate for } n = 6. \\
a_{10} &= 3072 \quad a_{10} = -3072
\end{align*}
\]

The 10th term is 3072 or -3072.

Find the 7th term of the geometric sequence with the given terms.

3a. \( a_4 = -8 \) and \( a_5 = -40 \)

3b. \( a_3 = 768 \) and \( a_4 = 48 \)

Geometric means are the terms between any two nonconsecutive terms of a geometric sequence.

If \( a \) and \( b \) are positive terms of a geometric sequence with exactly one term between them, the geometric mean is given by the following expression.

\[ \sqrt{ab} \]

Example 4
Finding Geometric Means

Find the geometric mean of \( \frac{1}{2} \) and \( \frac{1}{32}. \)

\[
\begin{align*}
\sqrt{ab} &= \sqrt{\left(\frac{1}{2}\right) \left(\frac{1}{32}\right)} \\
&= \sqrt{\frac{1}{64}} = \frac{1}{8} \quad \text{Use the formula.}
\end{align*}
\]

4. Find the geometric mean of 16 and \( 25. \)
The indicated sum of the terms of a geometric sequence is called a **geometric series**. You can derive a formula for the partial sum of a geometric series by subtracting the product of $S_n$ and $r$ from $S_n$ as shown:

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$$

$$-rS_n = -a_1 r - a_1 r^2 - \cdots - a_1 r^{n-1} - a_1 r^n$$

$$S_n - rS_n = a_1 - a_1 r^n$$

$$S_n (1 - r) = a_1 (1 - r^n)$$

$$S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right)$$

### Example

**Finding the Sum of a Geometric Series**

Find the indicated sum for each geometric series.

A. $S_5$ for $3 - 6 + 12 - 24 + \cdots$

Step 1: Find the common ratio.

$$r = \frac{a_2}{a_1} = \frac{-6}{3} = -2$$

Step 2: Find $S_5$ with $a_1 = 3$, $r = -2$, and $n = 7$.

$$S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right)$$

$$S_5 = 3 \left( \frac{1 - (-2)^7}{1 - (-2)} \right)$$

$$= 3 \left( \frac{1 - 128}{3} \right)$$

$$= 129$$

Check: Use a graphing calculator.

B. $\sum_{k=1}^{5} \left( \frac{1}{3} \right)^{k-1}$

Step 1: Find the first term.

$$a_1 = \left( \frac{1}{3} \right)^{0} = 1$$

Step 2: Find $S_5$.

$$S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right)$$

$$S_5 = 1 \left( \frac{1 - \left( \frac{1}{3} \right)^5}{1 - \left( \frac{1}{3} \right)} \right)$$

$$= \frac{1 - \left( \frac{1}{243} \right)}{\frac{2}{3}}$$

$$= \frac{242}{243} \approx 1.21$$

Check: Use a graphing calculator.
EXAMPLE 6

Sports Application

The Wimbledon Ladies’ Singles Championship begins with 128 players. The players compete until there is 1 winner. How many matches must be scheduled in order to complete the tournament?

Step 1 Write a sequence.

Let \( n \) = the number of rounds, \( a_n \) = the number of matches played in the \( n \)th round, and \( S_n \) = the total number of matches played through \( n \) rounds.

\[
a_n = 64 \left( \frac{1}{2} \right)^{n-1}
\]

The first round requires 64 matches, so \( a_1 = 64 \). Each successive match requires \( \frac{1}{2} \) as many, so \( r = \frac{1}{2} \).

Step 2 Find the number of rounds required.

\[
1 = 64 \left( \frac{1}{2} \right)^{n-1}
\]

Isolate the exponential expression by dividing by 64.

\[
\frac{1}{64} = \left( \frac{1}{2} \right)^{n-1}
\]

Express \( \frac{1}{64} \) as a power of \( \frac{1}{2} \):

\[
\left( \frac{1}{2} \right)^3 = \left( \frac{1}{2} \right)^{n-1}
\]

\[
6 = n - 1
\]

Solve for \( n \).

\[
7 = n
\]

Step 3 Find the total number of matches after 7 rounds.

\[
S_7 = 64 \left( 1 - \left( \frac{1}{2} \right)^7 \right) = 127
\]

Sum function for geometric series.

127 matches must be scheduled to complete the tournament.

6. Real Estate A 6-year lease states that the annual rent for an office space is $84,000 the first year and will increase by 8% each additional year of the lease. What will the total rent expense be for the 6-year lease? $616,218.04

THINK AND DISCUSS

1. Find the next three terms of the geometric sequence that begins 3, 6, …. Then find the next three terms of the arithmetic sequence that begins 3, 6, ….

2. Compare the geometric mean of 4 and 16 with the mean, or average.

3. GET ORGANIZED Copy and complete the graphic organizer. In each box, summarize your understanding of geometric sequences.
GUIDED PRACTICE

1. **Vocabulary** The term between two given terms in a geometric sequence is the _?_. (geometric mean or geometric series) _geometric mean_

   **See Example 1**
   p. 890

   Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.

   2. 
   \(-10, 12, -14, 16, \ldots\)  
   - arithmetic; \(d = -2\)  
   - geometric, \(r = \frac{6}{5}\)

   **See Example 2**
   p. 891

   Find the 10th term of each geometric sequence.

   5. 
   \(2, 6, 18, 54, \ldots\)  
   - geometric, \(r = 3\)

   **See Example 3**
   p. 892

   Find the 6th term of the geometric sequence with the given terms.

   8. 
   \(a_4 = -12, a_5 = -4\)  
   - geometric, \(r = 3\)

   **See Example 4**
   p. 892

   Find the geometric mean of each pair of numbers.

   11. 
   \(6 \text{ and } \frac{3}{8}\)

   **See Example 5**
   p. 893

   Find the indicated sum for each geometric series.

   14. 
   \[
   \sum_{k=1}^{5} (-3)^{k-1}
   \]  
   - geometric, \(r = \frac{3}{2}\)

   **See Example 6**
   p. 894

   18. **Salary** In his first year, a math teacher earned $32,000. Each successive year, he earned a 5% raise. How much did he earn in his 20th year? What were his total earnings over the 20-year period? $80,862.41; $1,058,110.53

PRACTICE AND PROBLEM SOLVING

Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.

19. 
\(-36, -49, -64, -81, \ldots\)
- arithmetic; \(d = -17\)

20. 
\(-2, -6, -18, -54, \ldots\)
- geometric; \(r = 3\)

21. 
\(2, 7, 12, 17, \ldots\)
- geometric; \(r = 3\)

22. 
\(\frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \frac{1}{250}, \frac{1}{1250}, \ldots\)
- geometric, \(r = \frac{1}{5}\)

23. 
\(-6, 12, -24, 48, \ldots\)
- arithmetic; \(d = 50\)

24. 
\(3200, 3000, 2000, 1000, \ldots\)
- geometric, \(r = \frac{1}{2}\)

25. 
\(12, 5, 24, 72, 216, \ldots\)
- geometric, \(r = \frac{1}{2}\)

26. 
\(a_4 = 54, a_5 = 162\)
- geometric, \(r = 3\)

27. 
\(a_3 = 13.5, a_6 = 20.25\)
- arithmetic; \(d = 500\)

28. 
\(a_4 = -4, a_6 = -100\)
- geometric, \(r = 5\)

29. 
\(9 \text{ and } \frac{1}{9}\)
- geometric, \(r = 3\)

30. 
\(18 \text{ and } 2\)
- geometric, \(r = 3\)

31. 
\(\frac{1}{5} \text{ and } 45\)
- arithmetic; \(d = 1\)

32. 
\(S_4 \text{ for } 1 + 5 + 25 + 125 + \ldots\)
- geometric, \(r = 5\)

33. 
\(S_5 \text{ for } 1 + 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots\)
- geometric, \(r = \frac{1}{5}\)

34. 
\(\sum_{k=1}^{5} \left(\frac{1}{3}\right)^{k-1}\)
- geometric, \(r = \frac{1}{3}\)

35. 
\(\sum_{k=1}^{10} (20)^{k-1}\)
- arithmetic; \(d = 20\)

12-4 Geometric Sequences and Series 895
36. **Genealogy** You have 2 biological parents, 4 biological grandparents, and 8 biological great-grandparents.

   a. How many direct ancestors do you have in the 6 generations before you? 12 generations? 126; 8190
   
   b. **What if...?** How does the explicit rule change if you are considered the first generation? The exponent changes from \( n \) to \( -1 \) such that \( a_n = 2^n \) becomes \( a_n = 2^{n-1} \).

Given each geometric sequence, (a) write an explicit rule for the sequence, (b) find the 10th term, and (c) find the sum of the first 10 terms.

37. \( 16 \cdot 8 \cdot 4 \cdot 2 \cdot \ldots \)

38. \( 4, 0.4, 0.04, 0.004, \ldots \)

39. \( 8, 16, 32, 64, \ldots \)

40. \(-22, -11, -\frac{11}{2}, -\frac{11}{4}, \ldots \)

41. \( 162, -54, 18, -6, \ldots \)

42. \( 12.5, 62.5, 312.5, 1562.5, \ldots \)

43. **Collectibles** Louis bought a vintage Rolling Stones concert shirt for $20. He estimates that the shirt will increase in value by 15% per year. How much is the shirt worth after 4 years? after 8 years? $34.98, $61.18

44. **College Tuition** New grandparents decide to pay for their granddaughter’s college education. They give the girl a penny on her first birthday and double the gift on each subsequent birthday. How much money will the girl receive when she is 18? 21? Will the money pay for her college education? Explain.

45. **Technology** You receive an e-mail asking you to forward it to 5 other people to ensure good luck. Assume that no one breaks the chain and that there are no duplications among the recipients. How many e-mails will have been sent after 10 generations, including yours, have received and sent the e-mail? 2,441,406

46. **Fractals** The Sierpinski carpet is a fractal based on a square. In each iteration, the center of each shaded square is removed.

   a. Given that the area of the original square is 1 square unit, write a sequence for the area of the \( n \)th iteration of the Sierpinski carpet.
   
   b. In which iteration will the area be less than \( \frac{1}{2} \) of the original area? 7th

47. **Paper** A piece of paper is 0.1 mm thick. When folded, the paper is twice as thick. Studies have shown that you can fold a piece of paper a maximum of 7 times. How thick will the paper be if it is folded on top of itself 7 times?

48. **Measurement** Several common U.S. paper sizes are shown in the table. You have 2 biological parents, 4 biological grandparents, and 8 biological great-grandparents.

   a. Given that the area of the original square is 1 square unit, write a sequence for the area of the \( n \)th iteration of the Sierpinski carpet.
   
   b. In which iteration will the area be less than \( \frac{1}{2} \) of the original area? 7th

    **47a. 12.8 mm**

49. **Common U.S. Paper Sizes**

<table>
<thead>
<tr>
<th>U.S. Paper Size</th>
<th>Dimensions (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (letter)</td>
<td>8 1/2 x 11</td>
</tr>
<tr>
<td>B (ledger)</td>
<td>11 x 17</td>
</tr>
<tr>
<td>C</td>
<td>17 x 22</td>
</tr>
<tr>
<td>D</td>
<td>22 x 34</td>
</tr>
<tr>
<td>E</td>
<td>34 x 44</td>
</tr>
</tbody>
</table>

49. **Fractals** The Sierpinski carpet is a fractal based on a square. In each iteration, the center of each shaded square is removed.

   a. Given that the area of the original square is 1 square unit, write a sequence for the area of the \( n \)th iteration of the Sierpinski carpet.
   
   b. In which iteration will the area be less than \( \frac{1}{2} \) of the original area? 7th

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   a. Given that the area of the original square is 1 square unit, write a sequence for the area of the \( n \)th iteration of the Sierpinski carpet.
   
   b. In which iteration will the area be less than \( \frac{1}{2} \) of the original area? 7th

    **47a. 12.8 mm**
49. This problem will prepare you for the Concept Connection on page 908. A movie earned $60 million in its first week of release and $9.6 million in the third week of release. The sales each week can be modeled by a geometric sequence.
   a. Estimate the movie's sales in its second week of release. $24 million
   b. By what percent did the sales decrease each week? 60%
   c. In what week would you expect sales to be less than $1 million? week 6
   d. Estimate the movie's total sales during its 8-week release period. about $99.93 million

50. **Biology** The population growth of bacteria in a petri dish each hour creates a geometric sequence. After 1 hour there were 4 bacteria cells, and after 5 hours there were 324 cells. How many cells were found at hours 2, 3, and 4?

51. **Critical Thinking** Find an arithmetic sequence, a geometric sequence, and a sequence that is neither arithmetic nor geometric that begins 1, 4, ..., 39, 95.

52. **Finance** Suppose that you pay $750 in rent each month. Suppose also that your rent is increased by 10% each year thereafter.
   a. Write a series that describes the total rent paid each year over the first 5 years, and find its sum.
   b. Use sigma notation to represent the series for the total rent paid each year over the first 10 years, and evaluate it.

53. **Music** The frequencies produced by playing C notes in ascending octaves make up a geometric sequence. C0 is the lowest C note audible to the human ear.
   a. The note commonly called middle C is C4. Find the frequency of middle C. about 261.6 Hz
   b. Write a geometric sequence for the frequency of C notes where n = 1 represents C1. \(a_n = 16.3(2)^n\)
   c. Humans cannot hear sounds with frequencies greater than 20,000 Hz. What is the first C note that humans cannot hear? C11

54. **Medicine** During a flu outbreak, a hospital recorded 16 cases the first week, 56 cases the second week, and 196 cases the third week.
   a. Write a geometric sequence to model the flu outbreak. \(a_n = 16(3)^{n-1}\)
   b. If the hospital did nothing to stop the outbreak, in which week would the total number infected exceed 10,000? week 6

55. **Graphing Calculator** Use the SEQ and SUM features to find each indicated sum of the geometric series 8 + 6 + 4.5 + ... to the nearest thousandth.
   a. \(S_5 = 30.188\)
   b. \(S_{20} = 31.899\)
   c. \(S_{50} = 31.994\)
   d. \(S_{100} = 32.000\)

56. **Critical Thinking** If a geometric sequence has \(r > 1\), what happens to the terms as \(n\) increases? What happens if \(0 < r < 1\)?

57. **Write About It** What happens to the terms of a geometric sequence when the first term is tripled? What happens to the sum of this geometric sequence?

58. Find the sum of the first 6 terms for the geometric series 4.5 + 9 + 18 + 36 + ...

<table>
<thead>
<tr>
<th>Scale of C's</th>
<th>Note</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C0</td>
<td>16.24</td>
</tr>
<tr>
<td></td>
<td>C1</td>
<td>32.7</td>
</tr>
<tr>
<td></td>
<td>C2</td>
<td>62.5</td>
</tr>
<tr>
<td></td>
<td>C3</td>
<td>130.8</td>
</tr>
<tr>
<td></td>
<td>C4</td>
<td></td>
</tr>
</tbody>
</table>

**Answers**

**Exercise 59**, for students who have difficulty with the material. Estimation shows that if a geometric sequence is applied to the model of the paper size, the shorter dimension becomes the longer one in each subsequent paper size, and the shorter dimension is doubled to become the longer dimension.

**b.** Possible answer: Each paper size has twice as much area as the previous size. The forms a geometric sequence with a common ratio of 2.

51. Possible answer: arithmetic: 1, 4, 7, 10, 13, ...; geometric: 1, 4, 16, 64, 256, ...; neither: 1, 4, 16, 25, ...

**Exercise 60**, for students who have difficulty with the material. Estimation shows that if a geometric sequence is applied to the model of the paper size, the shorter dimension becomes the longer one in each subsequent paper size, and the shorter dimension is doubled to become the longer dimension.

**b.** Possible answer: Each paper size has twice as much area as the previous size. The forms a geometric sequence with a common ratio of 2.

55e. Yes, the series appears to be approaching 32.

55f. Yes, the series appears to be approaching 32.

**Concept Connection**

Exercise 58, it may be helpful to write the 5th and 6th terms. Both A and B can be eliminated because the 6th term is 144 and the sum must be greater than this. Estimation shows that C is the correct choice.

For students who have difficulty with Exercise 59, remind them that the graph of a geometric sequence is similar in shape to the graph of an exponential function. Only choice G is close to this shape.

**Answers**

**48a.** Possible answer: The longer dimension becomes the shorter dimension in each subsequent paper size, and the shorter dimension is doubled to become the longer dimension.

**b.** Possible answer: Each paper size has twice as much area as the previous size. The forms a geometric sequence with a common ratio of 2.

52. Possible answer: arithmetic: 1, 4, 7, 10, 13, ...; geometric: 1, 4, 16, 64, 256, ...; neither: 1, 4, 16, 25, ...

55. Possible answer: For \(r > 1\), the sequence increases or decreases without bound as \(n\) increases. For \(0 < r < 1\), the sequence decreases toward a specific value as \(n\) increases.

57. Possible answer: Each term in the geometric sequence is tripled, and the sum of the series is also tripled.

**12-4 CHALLENGE**

12-4 Geometric Sequences and Series 897

**Exercise 61**, for students who have difficulty with the material. Estimation shows that if a geometric sequence is applied to the model of the paper size, the shorter dimension becomes the longer one in each subsequent paper size, and the shorter dimension is doubled to become the longer dimension.

**b.** Possible answer: Each paper size has twice as much area as the previous size. The forms a geometric sequence with a common ratio of 2.

55e. Yes, the series appears to be approaching 32.

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**Concept Connection**

Exercise 58, it may be helpful to write the 5th and 6th terms. Both A and B can be eliminated because the 6th term is 144 and the sum must be greater than this. Estimation shows that C is the correct choice.

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**48a.** Possible answer: The longer dimension becomes the shorter dimension in each subsequent paper size, and the shorter dimension is doubled to become the longer dimension.

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52. Possible answer: arithmetic: 1, 4, 7, 10, 13, ...; geometric: 1, 4, 16, 64, 256, ...; neither: 1, 4, 16, 25, ...

55. Possible answer: For \(r > 1\), the sequence increases or decreases without bound as \(n\) increases. For \(0 < r < 1\), the sequence decreases toward a specific value as \(n\) increases.

57. Possible answer: Each term in the geometric sequence is tripled, and the sum of the series is also tripled.
59. Which graph might represent the terms of a geometric sequence?

60. Find the first 3 terms of the geometric sequence with \(a_1 = -192\) and \(a_5 = -768\).
   - (A) \(-3, 6, 12\)
   - (B) \(-3, -12, -48\)

61. Which represents the sum of the series \(10 - 15 + 22.5 - 33.75 + 50.625\)?
   - (A) \(\sum_{k=1}^{5} 10 \left(\frac{3}{2}\right)^{k-1}\)
   - (B) \(\sum_{k=1}^{5} -10 \left(\frac{3}{2}\right)^{k-1}\)

62. Short Response Why does the general rule for a geometric sequence use \(n - 1\) instead of \(n\)? Explain.

### CHALLENGE AND EXTEND

#### Graphing Calculator
For each geometric sequence, find the first term with a value greater than 1,000,000.

- \(a_1 = 10\) and \(r = 2\)
- \(a_1 = \frac{1}{4}\) and \(r = 4\)
- \(a_1 = 0.01\) and \(r = 3.2\)

66. The sum of three consecutive terms of a geometric sequence is 73.5. If the product of these terms is 2744, what are the terms? \(3.5, 14, 56, 128\)

67. Consider the geometric sequence whose first term is 55 with the common ratio \(\frac{1 + \sqrt{5}}{2}\).
   - a. Find the next 5 terms rounded to the nearest integer. \(89, 144, 233, 377, 610\)
   - b. Add each pair of successive terms together. What do you notice?
   - c. Make a conjecture about this sequence.

Beginning with 55, this sequence appears to round to the Fibonacci numbers.

### SPIRAL REVIEW

Identify the zeros and asymptotes of each function. (Lesson 8-2)

- \(f(x) = \frac{x^2 + 2x - 3}{x + 1}\)
- \(f(x) = \frac{x + 5}{x^2 - x - 12}\)

71. Shopping During a summer sale, a store gives a 20% discount on all merchandise.
   - On Mondays, the store takes another 10% off of the sale price.
   - Find the cost on Monday of an item with an original price of \(x\) dollars. \(f(x) = 0.9(0.8x) = 0.72x\)
   - Find the cost on Monday of an item originally priced at \$275. \$198

Find the 10th term of each arithmetic sequence. (Lesson 12-3)

- \(72. 8, 65, 52, 39, 26, \ldots -39\)
- \(73. 1.7, 7.3, 12.9, 18.5, 24.1, \ldots 52.1\)
- \(74. 9.42, 9.23, 9.04, 8.85, 8.66, \ldots 7.71\)
- \(75. 16.4, 26.2, 36, 45.8, 55.6, \ldots 104.6\)

### Answers

62. Possible answer: The common ratio is not used to find the first term, so it is used only \(n - 1\) times to find the \(n\)th term.

63. \(a_{10} = 1,310,720\)
64. \(a_{12} = 1,048,576\)
65. \(a_{17} = 1,208,925.82\)

68. zeros: 1 and -3; vertical asymptote: \(x = -1\)

69. zero: -5; vertical asymptotes: \(x = -2\) and \(x = 3\); horizontal asymptote: \(y = 0\)

70. zeros: 4 and -4; vertical asymptote: \(x = 0\)
Explore Infinite Geometric Series

You can explore infinite geometric series by using a sequence of squares.

**Activity**

1. On a piece of graph paper, draw a $16 \times 16$ unit square. Note that its perimeter is 64 units.
2. Starting at one corner of the original square, draw a new square with side lengths half as long, or in this case, $8 \times 8$ units. Note that its perimeter is 32 units.
3. Create a table as shown at right. Fill in the perimeters and the cumulative sum of the perimeters that you have found so far.

<table>
<thead>
<tr>
<th>Square</th>
<th>Perimeter</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16 \times 16$</td>
<td>64</td>
<td>64</td>
</tr>
<tr>
<td>$8 \times 8$</td>
<td>32</td>
<td>96</td>
</tr>
<tr>
<td>$4 \times 4$</td>
<td>16</td>
<td>112</td>
</tr>
<tr>
<td>$2 \times 2$</td>
<td>8</td>
<td>120</td>
</tr>
<tr>
<td>$1 \times 1$</td>
<td>4</td>
<td>124</td>
</tr>
<tr>
<td>$\frac{1}{2} \times \frac{1}{2}$</td>
<td>2</td>
<td>126</td>
</tr>
</tbody>
</table>

**Try This**

1. Copy the table, and complete the first 6 rows.
2. Use summation notation to write a geometric series for the perimeters. Use the formula $\sum_{k=1}^{n} \frac{64}{1} \left(\frac{1}{2}\right)^{k-1}$
3. Use a graphing calculator to find the sum of the first 20 terms of the series. **Possible answer**: The sum of the perimeters is 128.
4. **Make a Conjecture**: Make a conjecture about the sum of the perimeter series if it were to continue indefinitely. **Possible answer**: The sum of the perimeters is about 127.99988.
5. Evaluate $\frac{64}{1 - \frac{1}{2}}$. How does this relate to your answer to Problem 4? **Possible answer**: The sum of the perimeters is 128; it is equal to the sum of the perimeters.
6. Copy and complete the table by finding the area of each square and the cumulative sums.

<table>
<thead>
<tr>
<th>Square</th>
<th>Area</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16 \times 16$</td>
<td>256</td>
<td>256</td>
</tr>
<tr>
<td>$8 \times 8$</td>
<td>64</td>
<td>320</td>
</tr>
<tr>
<td>$4 \times 4$</td>
<td>16</td>
<td>336</td>
</tr>
<tr>
<td>$2 \times 2$</td>
<td>4</td>
<td>340</td>
</tr>
<tr>
<td>$1 \times 1$</td>
<td>1</td>
<td>341</td>
</tr>
<tr>
<td>$\frac{1}{2} \times \frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$341 \frac{1}{3}$</td>
</tr>
</tbody>
</table>

7. **Draw a Conclusion**: Write a formula for the sum of an infinite geometric sequence. **Possible answer**: $S = \frac{a_1}{1 - r}$

**Close**

**Key Concept**

When $|r| < 1$, a formula can be used to find the sum of an infinite geometric sequence.

**Assessment**

**Journal**: Have students explain why $|r| < 1$ must be true to find the sum of an infinite geometric sequence.
Mathematical Induction and Infinite Geometric Series

**Objectives**
Find sums of infinite geometric series.
Use mathematical induction to prove statements.

**Vocabulary**
infinite geometric series
converge
limit
diverge
mathematical induction

**Why learn this?**
You can use infinite geometric series to explore repeating patterns. (See Exercise 58.)

In Lesson 12-4, you found partial sums of geometric series. You can also find the sums of some infinite geometric series. An **infinite geometric series** has infinitely many terms. Consider the two infinite geometric series below.

\[ S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \]
\[ R_n = \frac{1}{32} + \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \cdots \]

Notice that the series \( S_n \) has a common ratio of \( \frac{1}{2} \) and the partial sums get closer and closer to 1 as \( n \) increases. When \( |r| < 1 \) and the partial sum approaches a fixed number, the series is said to **converge**. The number that the partial sums approach, as \( n \) increases, is called a **limit**.

For the series \( R_n \), the opposite applies. Its common ratio is 2, and its partial sums increase toward infinity. When \( |r| \geq 1 \) and the partial sum does not approach a fixed number, the series is said to **diverge**.

**California Standards**

**Algebra 2 21.0**
Also covered: **22.0**

**Students find the general term and the sums of arithmetic series and of both finite and infinite geometric series.**

**23.0**
**Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.**

**Math Humor**

**Cat Theorem:** A cat has nine tails.

**Proof:** No cat has eight tails. A cat has one tail more than no cat. Therefore, a cat has nine tails.

**EXAMPLE 1**

**Finding Convergent or Divergent Series**

Determine whether each geometric series converges or diverges.

**A**
\[ 20 + 24 + 28.8 + 34.56 + \cdots \]
\[ r = \frac{24}{20} = 1.2, \quad |r| \geq 1 \]
The series diverges and does not have a sum.

**B**
\[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots \]
\[ r = \frac{\frac{1}{3}}{1} = \frac{1}{3}, \quad |r| < 1 \]
The series converges and has a sum.

**Explore**

Determine whether each geometric series converges or diverges.

**1a.**
\[ \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \cdots \]
**diverges**

**1b.**
\[ 32 + 16 + 8 + 4 + 2 + \cdots \]
**converges**

**Exploration Questions**

1. Describe what happens to the partial sums of the sequence as you add more and more terms.

**Explorations and answers are provided in Alternate Openers: Explorations Transparencies.**

**12-5 Organizer**

**Objectives:** Find sums of infinite geometric series.
Use mathematical induction to prove statements.

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**Warm Up**

Evaluate.

1. \( \frac{3}{4} \) \( \frac{1}{10} \)
2. \( \frac{7}{10} \) \( \frac{1}{0.3} \) \( \frac{10}{1} \)
3. Write 0.6 as a fraction in simplest form. \( \frac{3}{5} \)
4. Find the indicated sum for the geometric series
\[ \sum_{k=1}^{7} 3(-1)^{k-1} \]
\[ = 3 \]

Also available on transparency

**Math Humor**

**Cat Theorem:** A cat has nine tails.

**Proof:** No cat has eight tails. A cat has one tail more than no cat. Therefore, a cat has nine tails.
If an infinite series converges, we can find the sum. Consider the series
\[ S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \]
from the previous page. Use the formula for the partial sum of a geometric series with \( a_1 = \frac{1}{2} \) and \( r = \frac{1}{2} \).

\[ S_n = a_1 \left( \frac{1-r^n}{1-r} \right) = \frac{1}{2} \left( \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \right) = \frac{1}{2} \left( \frac{1-\left(\frac{1}{2}\right)^n}{\frac{1}{2}} \right) = \frac{1}{2} - \left(\frac{1}{2}\right)^n \]

Graph the simplified equation on a graphing calculator. Notice that the sum levels out and converges to 1.

As \( n \) approaches infinity, the term \( \left(\frac{1}{2}\right)^n \) approaches zero. Therefore, the sum of the series is 1. This concept can be generalized for all convergent geometric series and proved by using calculus.

### Sum of an Infinite Geometric Series

The sum of an infinite geometric series \( S \) with common ratio \( r \) and \( |r| < 1 \) is
\[ S = \frac{a_1}{1-r} \]

where \( a_1 \) is the first term.

#### Example 2

**Finding the Sums of Infinite Geometric Series**

Find the sum of each infinite geometric series, if it exists.

**A.**
\[ 5 + 4 + 3.2 + 2.56 + \cdots \]
\[ r = 0.8 \]  
**Converges:** \( |r| < 1 \)
\[ S = \frac{a_1}{1-r} \]  
**Sum formula**
\[ S = \frac{5}{1-0.8} = \frac{5}{0.2} = 25 \]

**Check**  
Graph \( y = S = \frac{1-(0.8)^n}{1-0.8} \) on a graphing calculator. The graph approaches \( y = 25 \).

**B.**
\[ \sum_{k=1}^{\infty} \frac{2}{3^k} \]
\[ \sum_{k=1}^{\infty} \frac{2}{3^k} = \frac{2}{1} - \frac{2}{3} + \frac{2}{9} + \cdots \]  
**Evaluate.**
\[ r = \frac{2}{3} \]  
**Converges:** \( |r| < 1 \)
\[ S = \frac{a_1}{1-r} = \frac{2}{1-\frac{2}{3}} = \frac{2}{\frac{1}{3}} = 6 \]

**Check**  
Graph \( y = S = \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} \) on a graphing calculator. The graph approaches \( y = 6 \).

Find the sum of each infinite geometric series, if it exists.

**2a.**
\[ 25 - 5 + 1 - \frac{1}{5} + \frac{1}{25} + \cdots \]
\[ 2b. \sum_{k=1}^{\infty} \frac{2}{3} \]

**Why does a graph help you check the sum of an infinite geometric series?**

**Example 1**

Determine whether each geometric series, converges or diverges.

A. **10 + 1 + 0.1 + 0.01 + \cdots** converges
B. **4 + 12 + 36 + 108 + \cdots** diverges

**Example 2**

Find the sum of each infinite geometric series, if it exists.

A. **1 - 0.2 + 0.04 - 0.008 + \cdots 0.83**
B. **\[ \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k - 1 \]**

Also available on transparency

**Common Error Alert**

Students may attempt to find the sum of an infinite series before knowing that the series converges. Remind students that only convergent geometric series have a finite sum.

**Power Presentations with PowerPoint**

**Additional Examples**

**Example 1**

Determine whether each geometric series, converges or diverges.

A. **10 + 1 + 0.1 + 0.01 + \cdots** converges
B. **4 + 12 + 36 + 108 + \cdots** diverges

**Example 2**

Find the sum of each infinite geometric series, if it exists.

A. **1 - 0.2 + 0.04 - 0.008 + \cdots 0.83**
B. **\[ \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k - 1 \]**

Also available on transparency

**INTERVENTION Questioning Strategies**

**EXAMPLE 1**

- What do you look for in the graph of an infinite geometric series to help you determine whether it converges? Explain.

**EXAMPLE 2**

- How does a graph help you check the sum of an infinite geometric series?
You can use infinite series to write a repeating decimal as a fraction.

### Example 3

**Writing Repeating Decimals as Fractions**

Write 0.232323... as a fraction in simplest form.

**Step 1** Write the repeating decimal as an infinite geometric series.

\[ 0.232323... = 0.23 + 0.0023 + 0.000023 + \cdots \]

*Use the pattern for the series.*

**Step 2** Find the common ratio.

\[ r = \frac{0.0023}{0.23} = \frac{1}{100} \text{ or } 0.01 \]

*|r| < 1; the series converges to a sum.*

**Step 3** Find the sum.

\[ S = \frac{a_1}{1 - r} \]

\[ = \frac{0.23}{1 - 0.01} = \frac{0.23}{0.99} = \frac{23}{99} \]

**Check** Use a calculator to divide the fraction $\frac{23}{99}$. ✔

### Example 4

3. Write 0.111... as a fraction in simplest form. $\frac{1}{9}$

You have used series to find the sums of many sets of numbers, such as the first 100 natural numbers. The formulas that you used for such sums can be proved by using a type of mathematical proof called **mathematical induction**.

### Proof by Mathematical Induction

To prove that a statement is true for all natural numbers $n$,

1. **Step 1** The base case: Show that the statement is true for $n = 1$.
2. **Step 2** Assume that the statement is true for a natural number $k$.
3. **Step 3** Prove that the statement is true for the natural number $k + 1$.

### Example 4

**Proving with Mathematical Induction**

Use mathematical induction to prove that the sum of the first $n$ natural numbers is $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$.

**Step 1** Base case: Show that the statement is true for $n = 1$.

\[ 1 = \frac{n(n + 1)}{2} = \frac{1(1 + 1)}{2} = \frac{2}{2} = 1 \]

*The base case is true.*

**Step 2** Assume that the statement is true for a natural number $k$.

\[ 1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2} \]

*Replace $n$ with $k$.*

### Additional Examples

**Example 3**
Write 0.63 as a fraction in simplest form. $\frac{7}{11}$

**Example 4**
Use mathematical induction to prove that $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}$.

1. $1 = \frac{1(3 - 1)}{2} = 1$.
2. $1 + 4 + \cdots + (3k - 2) = \frac{k(3k - 1)}{2}$.
3. Add $3(k + 1) - 2$ to each side.

\[ 1 + \cdots + 3k - 2 + 3(k + 1) - 2 \]

\[ = \frac{k(3k - 1)}{2} + 3(k + 1) - 2 \]

\[ = \frac{2k(3k - 1)}{2} + 6(k + 1) - 4 \]

\[ = \frac{3k^2 - k + 6k + 2}{2} \]

\[ = \frac{3k^2 + 5k + 2}{2} \]

\[ = \frac{(k + 1)(3k + 2)}{2} \]

\[ = \frac{(k + 1)(3k + 3 - 1)}{2} \]

\[ = \frac{(k + 1)[3(k + 1) - 1]}{2} \]

Also available on transparency
Step 3 Prove that it is true for the natural number \( k + 1 \).

\[
1 + 2 + \cdots + k = \frac{k(k + 1)}{2}.
\]

\[
1 + 2 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)
\]

Add the next term \( (k + 1) \) to each side.

\[
= \frac{k(k + 1) + 2(k + 1)}{2}
\]

Find the common denominator.

\[
= \frac{k(k + 1) + 2(k + 1)}{2}
\]

Add numerators.

\[
= \frac{(k + 1)(k + 2)}{2}
\]

Factor out \( k + 1 \).

\[
= \frac{(k + 1)[(k + 1) + 1]}{2}
\]

Write with \( k + 1 \).

Therefore, \( 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \).

4. Use mathematical induction to prove that the sum of the first \( n \) odd numbers is \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \).

Mathematical statements that seem to be true may in fact be false. By finding a counterexample, you can disprove a statement.

**EXAMPLE 5**

**Using Counterexamples**

Identify a counterexample to disprove \( 2^n \geq n^2 \), where \( n \) is a real number.

\[
\begin{align*}
2^0 & \geq (0)^2 \hspace{1cm} \checkmark \\
2^1 & \geq (1)^2 \hspace{1cm} \checkmark \\
2^2 & \geq (4)^2 \hspace{1cm} \checkmark \\
2^3 & \geq (-1)^2 \hspace{1cm} x
\end{align*}
\]

\( 2^n \geq n^2 \) is not true for \( n = -1 \), so it is not true for all real numbers.

5. Identify a counterexample to disprove \( a^2 \geq 2a + 1 \), where \( a \) is a real number.

when \( a = 5 \):

\[
\frac{5^2}{2} \geq 2(5) + 1
\]

\( 12.5 \leq 11 \)

**THINK AND DISCUSS**

1. Explain how to determine whether a geometric series converges or diverges.

2. Explain how to represent the repeating decimal \( 0.8\overline{3} \) as an infinite geometric series.

3. **GET ORGANIZED** Copy and complete the graphic organizer.

<table>
<thead>
<tr>
<th>Example</th>
<th>Common Ratio</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convergent Series</td>
<td>( a &lt; 1 )</td>
<td>( \frac{a^n}{1-a} )</td>
</tr>
<tr>
<td>Divergent Series</td>
<td>( a \geq 1 )</td>
<td>Diverges</td>
</tr>
</tbody>
</table>

**Answers to Think and Discuss**

Possible answers:

1. Find the common ratio. If \( |r| < 1 \), the series converges. If \( |r| \geq 1 \), the series diverges.

2. Write \( 0.8\overline{3} \) as \( \frac{8}{9} \) plus the infinite geometric series \( 0.03 + 0.003 + \cdots \).

**12.5 Exercises**

### 12.5 Practice A

Determine whether each geometric series converges or diverges.

1. $1 \quad 2. \quad 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots \quad \text{converges}$
2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \quad \text{converges}$
3. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots \quad \text{converges}$
4. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \quad \text{converges}$

**12.5 Practice B**

Determine whether each geometric series converges or diverges.

1. $\frac{3}{4} \quad 2. \quad \frac{3}{4} + \frac{1}{2} + \frac{3}{4} + 2 + \cdots \quad \text{diverges}$
3. $\frac{3}{4} \quad 2. \quad \frac{3}{4} + \frac{1}{2} + \frac{3}{4} + 2 + \cdots \quad \text{diverges}$
4. $\frac{3}{4} \quad 2. \quad \frac{3}{4} + \frac{1}{2} + \frac{3}{4} + 2 + \cdots \quad \text{diverges}$

**Guided Practice**

1. **Vocabulary**
   - An infinite geometric series whose sum approaches a fixed number is said to converge.
   - A geometric series whose sum diverges.

2. **Examples**
   - Find the common ratio and sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the common ratio and sum of the first 4 terms of the geometric series $a_n = (5)^n$.

3. **Guided Practice**
   - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

4. **Guided Practice**
   - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

5. **Guided Practice**
   - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

6. **Guided Practice**
   - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

7. **Guided Practice**
   - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

8. **Guided Practice**
   - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

9. **Guided Practice**
   - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
   - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

10. **Guided Practice**
    - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
    - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

11. **Guided Practice**
    - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
    - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

12. **Guided Practice**
    - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
    - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

13. **Guided Practice**
    - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
    - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.

14. **Guided Practice**
    - Find the sum of the first 4 terms of the geometric series $a_n = (3)^n$.
    - Find the sum of the first 4 terms of the geometric series $a_n = (5)^n$.
29. **Art** Ojos de Dios are Mexican holiday decorations. They are made of yarn, which is wrapped around sticks in a repeated square pattern. Suppose that the side length of the outer square is 8 inches. The side length of each inner square is 90% of the previous square’s length. How much yarn will be required to complete the decoration? (Assume that the pattern is represented by an infinite geometric series.)

Find the sum of each infinite geometric series, if it exists.

30. $215 - 86 + 34.4 - 13.76 + \cdots$

31. $500 + 400 + 320 + \cdots$

32. $8 - 10 + 12.5 - 15.625 + \cdots$

33. $\sum_{k=1}^{\infty} -5\left(\frac{1}{2}\right)^{k-1}$

34. $\sum_{k=1}^{\infty} \frac{2}{3}\left(\frac{1}{4}\right)^{k-1}$

35. $\sum_{k=1}^{\infty} \frac{5}{3}\left(\frac{1}{2}\right)^{k-1}$

36. $-25 - 30 - 36 - 43.2 + \cdots$

37. $\sum_{k=1}^{\infty} 200\left(0.6\right)^{k-1}$

38. **Geometry** A circle of radius $r$ has smaller circles drawn inside it as shown. Each smaller circle has half the radius of the previous circle.

   a. Write an infinite geometric series in terms of $r$ that expresses the circumferences of the circles, and find its sum.

   b. Find the sum of the circumferences for the infinite set of circles if the first circle has a radius of 3 cm. $12\pi \approx 37.68$ cm

39. Write each repeating decimal as a fraction in simplest form.

   39. 0.7 $\overline{3}$

   40. 0.9 $\overline{1}$

   41. 0.123 $\overline{41}$

   42. 0.18 $\overline{2}$

   43. 0.5 $\overline{9}$

   44. 0.054 $\overline{2}$

45. **Music** Due to increasing online downloads, CD sales have declined in recent years. Starting in 2001, the number of CDs shipped each year can be modeled by a geometric sequence.

   a. Estimate the number of CDs that will be shipped in 2010. About 415.0 million

   b. Estimate the total number of CDs shipped from 2001 through 2010. About 6.2 billion

   c. Suppose that the geometric series continued indefinitely. Find the total number of CDs shipped from 2001. About 11 billion

Use mathematical induction to prove each statement.

46. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$

47. $1 + 4 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

48. $1^2(2) + 2^2(3) + 3^2(4) + \cdots + n^2(n+1) = \frac{n(n+1)(n+2)(n+3)}{3}$

49. $\frac{1}{2} + 2 + 4 + \cdots + \frac{1}{8} + \frac{1}{4} + \cdots = 1 - \left(\frac{1}{2}\right)^n$

### Additional Exercises

47–49. See p. A47.
50. This problem will prepare you for the Concept Connection on page 908.

A movie earned $80 million in the first week that it was released. In each successive week, sales declined by about 40%.

a. Write a general rule for a geometric sequence that models the movie's sales each week. $a_n = 80(0.6)^{n-1}$, where $a_n$ represents millions of dollars.
b. Estimate the movie's total sales in the first 6 weeks. about $190.67$ million.
c. If this pattern continued indefinitely, what would the movie's total sales be? $200$ million.

51. **Game Shows** Imagine that you have just won the grand prize on a game show. You can choose between two payment options as shown. Which would you choose, and why?

Identify a counterexample to disprove each statement, where $x$ is a real number.

52. $\frac{x^4}{3} \leq 2x = -1$

53. $x^4 - 1 \geq 0 \ x = \frac{1}{2}$

54. $\ln x^2 = \ln x = 0$

55. $2x^2 \leq 3x^3 = -2$

56. $2x^2 - x \leq 0 \ x = \frac{1}{4}$

57. $12x - x^2 > 25 \ x = 0$

58. **Geometry** The midpoints of the sides of a 12-inch square are connected to form another concentric square as shown. Suppose that this process is continued without end to form a sequence of concentric squares.

a. Find the perimeter of the 2nd square. $24\sqrt{2}$ is about $33.94$ in.

b. Find the sum of the perimeters of the squares.

c. Find the sum of the areas of the squares. $288\text{ in}^2$

d. Write the sum of the perimeters in summation notation for the general case of a square with side length $s$. Then write the sum of the areas for the general case.

e. Which series decreases faster, the sum of the perimeters or the sum of the areas? How do you know?

59. **Critical Thinking** Compare the partial-sum $S_n$ with the sum $S$ for an infinite geometric series when $a_1 > 0$ and $r$ is less than 1. Which is greater? What if $a_1 < 0$?

60. **Write About It** Why might the notation for a partial sum $S_n$ change for the sum $S$ for an infinite geometric series? The subscript $n$ refers to a finite number of terms, but an infinite geometric series has an infinite number of terms.

### 12.5 READING STRATEGIES

#### Infinite geometric series can be converging or diverging. To identify if an infinite geometric series is converging or diverging, first identify the common ratio. A series converges if the common ratio is less than 1. A series diverges if the common ratio is greater than or equal to 1.

**Akhbar walking:**
- Decide whether the graph is a diverging series. What happens when $x$ approaches negative infinity?
- Decide whether the graph is a converging series. What happens when $x$ approaches positive infinity?

**Possible answer:** A diverging series increases in the number of terms. A converging series approaches a fixed number as the number of terms increases.

**Exercise 58:**
- Identify the common ratio of the series.
- Determine whether the series is converging or diverging. The $100$ million the first year and half of the previous year’s amount for eternity

#### PRISE PAYMENT OPTIONS

<table>
<thead>
<tr>
<th>A</th>
<th>$1$ million the first year and half of the previous year’s amount for eternity</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$100,000$ a year for 20 years</td>
</tr>
</tbody>
</table>

### 12.5 REVIEW FOR MASTERY

#### Determine whether each geometric series converges or diverges. Compare the absolute value of the common ratio with 1. If the common ratio is greater than 1, the series diverges. If the common ratio is less than 1, the series converges.

- **Converging:** $|r| < 1$
- **Diverging:** $|r| \geq 1$

#### Example 1:
- $a_n = 3(\frac{2}{3})^n$
- $r = \frac{2}{3}$
- **Converging:** $|r| < 1$

#### Example 2:
- $a_n = -2(5)^n$
- $r = 5$
- **Diverging:** $|r| \geq 1$
64. Which graph represents a converging infinite geometric series?

65. Extended Response Use mathematical induction to prove

\[3 + 5 + \cdots + (2n + 1) = n(n + 2)\]. Show all of your work.

66. Write each repeating decimal as a fraction in simplest form.

67. 0.166666...

68. 0.416666...

69. Critical Thinking Can an infinite arithmetic series approach a limit like an infinite geometric series? Explain why or why not.

70. Geometry Consider the construction that starts with a 12-inch square and contains concentric squares as indicated. Notice that a spiral is formed by the sequence of segments starting at a corner and moving inward as each midpoint is reached. A second similar spiral determines the area shown in blue.

a. Use the sum of a series to find the length of the spiral indicated in red. 12 + 6 \sqrt{2} = 20.485 in.

b. Use the sum of a series to find the polygonal area indicated in blue. 36 in^2.

c. Is your answer to the sum of the polygonal area in part b reasonable? Explain. Yes; the area of the spiral is \(\frac{3}{4}\) the area of the largest square.

71. Football A kickoff specialist kicks 80% of his kickoffs into the end zone. What is the probability that he kicks at least 4 out of 5 of his next kickoffs into the end zone? (Lesson 11-6) 73.728%

72. Geometry Consider the pattern of figures shown. (Lesson 12-3)

\[a_n = 3 + 2(n - 1)\]

a. Find the number of dots in each of the next 3 figures in the pattern. 9, 11, 13

b. Write a general rule for the sequence of the number of dots in the nth figure.

c. How many dots will be in the 22nd figure in the pattern? 45 dots

Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference. (Lesson 12-4)

73. 297, 99, 33, 11, ... geometric; \(r = \frac{1}{3}\)

74. 4, 8, 16, ... arithmetic; \(d = \frac{4}{3}\)

75. 25, 100, 250, 1000, ... neither

76. 4, 8.5, 7.6, 6.912, ... geometric; \(r = \frac{5}{6}\)

12-5 Mathematical Induction and Infinite Geometric Series

12-5 Challenge

Solve each equation.

1. Determine whether the geometric series
   150 + 30 + 6 + ... converges or diverges, and find the sum if it exists. converges; 187.5

2. Write \(0.0044\) as a fraction in simplest form. \(\frac{1}{250}\)

3. Either prove by induction or provide a counterexample to disprove the following statement: \(1 + 2 + 3 + 4 + \cdots + n = \frac{n^2 + 1}{2}\) counterexample: \(n = 2\)

Also available on transparency
Exploring Geometric Sequences and Series

Sticky Business  Big-budget movies often have their greatest sales in the first weekend, and then weekend sales decrease with each passing week. After a movie has been released for a few weeks, movie studios may try to predict the total sales that the movie will generate.

1. Find the ratios of the sequences of weekend sales for Spider-Man and Spider-Man 2.

2. Write the rule for a geometric sequence that could be used to estimate the sales for Spider-Man in a given weekend.

3. Use the sequence from Problem 2 to predict Spider-Man's weekend sales for weeks 4 and 5.

4. Write and evaluate a series in summation notation to find Spider-Man's total weekend sales for the first 5 weekends of its release.

5. Suppose that the series from Problem 4 continued infinitely. Estimate the total weekend sales for Spider-Man. The actual total weekend sales for Spider-Man were about $311.1 million. How does this compare with your estimate? About $306.4 million; the estimate is very close.

6. Would a geometric sequence be a good model for the weekend sales of Spider-Man 2? Justify your answer.

Weekend Box Office Sales (million $)

<table>
<thead>
<tr>
<th>Weekend</th>
<th>Spider-Man</th>
<th>Spider-Man 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>114.9</td>
<td>115.8</td>
</tr>
<tr>
<td>2</td>
<td>71.4</td>
<td>45.2</td>
</tr>
<tr>
<td>3</td>
<td>45.0</td>
<td>24.8</td>
</tr>
</tbody>
</table>

\[
\sum_{k=1}^{5} 114.9(0.625)^{k-1}; \quad \text{about } $277.2 \text{ million}
\]

Answers

1. \(a_n = 0.62, \approx 0.63; \quad a_n = 0.39, \approx 0.55\)

2. \(a_n = 114.9(0.625)^{n-1}, \text{ where } a_n \text{ represents sales in millions of dollars}\)

6. Possible answer: No; the ratios for Spider-Man 2 are not close to being constant, so it is very hard to develop an accurate geometric sequence.

INTERVENTION

Scaffolding Questions

1. How do you find common ratios? Divide each term by the previous term.

2. What must you know to write the rule for a geometric sequence? the first term and the common ratio

3. How can you find the 4th and 5th terms? Evaluate the series for \(n = 4\) and \(n = 5\).

4. What is the formula for the partial sum of a geometric series?

\[
S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right)
\]

5. What is the formula for the sum of an infinite geometric series?

\[
S_n = \frac{a_1}{1 - r}
\]

6. What identifies a geometric sequence? common ratio

Extension

Have students find the box office data for a recent hit movie and determine whether a geometric sequence would be a good model for its weekend sales. Check students' work.
1. Write an infinite geometric series to represent the distance that the ball travels after it initially hits the ground. 

2. How much in total will be paid for electricity over the first 6 years?

3. Find the geometric mean of each pair of numbers.

4. Use mathematical induction to prove \( 4^n - 1 \) is divisible by 3 for all positive integers n.

5. Find the indicated sum for each geometric series.

6. Find the sum of each infinite series, if it exists.

7. Find the 10th term of each geometric sequence with the given terms.

8. Find the total distance that the ball travels after it initially hits the ground. (Hint: The ball travels up and down on each bounce.)

9. Of its previous height after each bounce.

10. How much will the business pay for electricity in the 6th year?

11. The cost for electricity is expected to rise at an annual rate of 8%. In its first year, a business spends $3000 for electricity.

12. a. How much will the business pay for electricity in the 6th year? $4407.98

b. How much in total will be paid for electricity over the first 6 years? $22,007.79

13. Find the sum of each series.

14. Use mathematical induction to prove \( 4 + 8 + 12 + \cdots + 4n = 2n(n + 1) \).

15. A table-tennis ball is dropped from a height of 5 ft. The ball rebounds to 60% of its previous height after each bounce.

a. Write an infinite geometric series to represent the distance that the ball travels after it initially hits the ground. (Hint: The ball travels up and down on each bounce.)

b. What is the total distance that the ball travels after it initially hits the ground? 15 ft
Finding the area under a curve is an important topic in higher mathematics, such as calculus. You can approximate the area under a curve by using a series of rectangles as shown in Example 1.

**Example 1**

**Finding Area Under a Curve**

Estimate the area under the curve \( f(x) = \frac{-3}{2}x^2 + 2x + 3 \) over \( 0 \leq x \leq 5 \).

Graph the function. Divide the area into 5 rectangles, each with a width of 1 unit.

Find the height of each rectangle by evaluating the function at the center of each rectangle, as shown in the table.

Approximate the area by finding the sum of the areas of the rectangles.

\[
A \approx \sum_{k=1}^{5} f\left(\frac{x_k + x_{k+1}}{2}\right) \cdot \frac{1}{2}
\]

The estimate of 20 square units is very close to the actual area of \( \frac{1919}{24} \) square units, which can be found by using calculus.

1. Estimate the area under the curve \( f(x) = -x^2 + 5x + 5.75 \) over \( 0 \leq x \leq 6 \). Use 6 intervals.

**Example 2**

**Finding Area Under a Curve by Using a Series**

Use the sum of a series to estimate the area under the curve \( f(x) = -x^2 + 50x \) over \( 0 \leq x \leq 50 \).

Step 1 Graph the function.

Step 2 Divide the area into 5 rectangles, each with a width of 10 units.

You can formalize the procedure for finding the area under the curve of a function by using the sum of a series.
Step 3  Find the value of the function at the center of each rectangle, as shown in the table.

Step 4  Write the sum that approximates the area.

\[
A = 10 \sum_{i=1}^{10} f(a_i) = 10 \left[ f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \right]
\]

\[
= 10 \left[ (225) + (525) + (625) + (525) + (225) \right]
\]

\[
= 10(2125) = 21,250
\]

The estimated area is 21,250 square units.

2. Use the sum of a series to estimate the area under the curve.

\[ f(x) = -x^2 + 9x + 3 \text{ over } 0 \leq x \leq 9 \]

Use 3 intervals.

\[ 155.25 \text{ square units} \]

Estimate the area under each curve. Use 4 intervals.

1. \[ f(x) = -\frac{1}{2}x^2 + 12x + 2 \text{ over } 0 \leq x \leq 24 \]

\[ 1236 \text{ square units} \]

2. \[ f(x) = -x^2 + 8x + 4 \text{ over } 0 \leq x \leq 8 \]

\[ 120 \text{ square units} \]

3. \[ f(x) = -\frac{1}{4}x^2 + 5x \text{ over } 0 \leq x \leq 16 \]

\[ 304 \text{ square units} \]

4. \[ f(x) = -0.1x^2 + 20x \text{ over } 0 \leq x \leq 200 \]

\[ 137,500 \text{ square units} \]

Use the sum of a series to estimate the area under each curve. Use 5 intervals.

5. \[ f(x) = -x^2 + 10x + 5 \text{ over } 0 \leq x \leq 10 \]

\[ 220 \text{ square units} \]

6. \[ f(x) = -x^2 + 30x \text{ over } 0 \leq x \leq 20 \]

\[ 1236 \text{ square units} \]

7. \[ f(x) = -\frac{1}{10}x^2 + 400 \text{ over } 0 \leq x \leq 50 \]

\[ 3360 \text{ square units} \]

8. \[ f(x) = -0.2x^2 + 28x + 300 \text{ over } 0 \leq x \leq 150 \]

\[ 15,875 \text{ square units} \]

9. **Physics** The graph shows a car’s speed versus time as the car accelerates. This realistic curve can be approximated by

\[ v(t) = -0.1t^2 + 7.3t \]

where \( v \) is the velocity in feet per second and \( t \) is the time in seconds.

\[ \text{a. Estimate the area under the curve for } 0 \leq t \leq 35.} \]

\[ \text{b. What does the area under the curve represent? Explain.} \]

(Hint: Consider the units of your answer to part a.)

10. **Energy Conservation** Daily electricity use peaks in the early afternoon and can be approximated by a parabola. Suppose that the rate of electricity use in kilowatts (kW) is modeled by the function

\[ f(x) = -1.25x^2 + 30x + 700 \]

where \( x \) represents the time in hours.

\[ \text{a. Write a sum to represent the area under the curve for a domain of } 0 \leq x \leq 24.} \]

\[ \text{b. Estimate the area under the curve.} 19,720 \text{ square units} \]

11. **Write About It** Explain how the units of the values on the \( x \)-axis and the units of the functions on the \( y \)-axis can be used to find the units that apply to the area under a curve.

12. **Critical Thinking** For a given function and domain, how would increasing the number of rectangles affect the approximation of the area under the curve?

Answers

9a. about 3056 square units
b. Distance in feet traveled in \( t \) s; to find the area of each rectangle, you multiply speed (ft/s) by time (s), and the result is a distance (ft).

10a. Possible answer: \[ A = 4 \sum_{i=1}^{6} f(\theta_i) \]

\[ = 4(755 + 835 + 875 + 875 + 835 + 755) \]

11. Possible answer: Because you find area by multiplying the length and the width, you multiply the units to find the units that apply to the area under the curve.

12. Possible answer: Increasing the number of rectangles would make the answer more accurate.

Students may be confused by the units of the answer in Exercises 9 and 10. Explain that the graphs are only abstract representations of a function and that the area under the curve is not measured in common units of area, such as square feet.
Complete the sentences below with vocabulary words from the list above.

1. A(n) __________ has a common difference, and a(n) __________ has a common ratio.
2. A series that has no limit __________, whereas a series that approaches a limit __________.
3. A(n) __________ defines the nth term. A(n) __________ defines the next term by using one or more of the previous terms.
4. A(n) __________ continues without end, and a(n) __________ has a last term.
5. Each step in a repeated process is called a(n) __________.

12-1 Introduction to Sequences (pp. 862–868)

Find the first 5 terms of the sequence with

\[ a_1 = -52; a_n = 0.5a_{n-1} + 2. \]

Evaluate the rule using each term to find the next term.

\[ \begin{array}{c|ccccc}
 n & 1 & 2 & 3 & 4 & 5 \\
 a_n & -52 & -24 & -10 & -3 & 0.5 \\
\end{array} \]

Write an explicit rule for the nth term of 100, 72, 44, 16, -12, ... .

Examine the differences or ratios.

Terms 100 72 44 16 -12
1st differences 28 28 28 28

The first differences are constant, so the sequence is linear.

The first term is 100, and each term is 28 less than the previous term.

The explicit rule is \( a_n = 100 - 28(n - 1). \)

Find the first 5 terms of each sequence.

6. \( a_n = n - 9 \)
7. \( a_n = \frac{1}{2}n^2 \)
8. \( a_n = \left(-\frac{3}{2}\right)^{n-1} \)
9. \( a_1 = 55 \) and \( a_n = a_{n-1} - 2 \)
10. \( a_1 = 200 \) and \( a_n = \frac{1}{5}a_{n-1} \)
11. \( a_1 = -3 \) and \( a_n = -3a_{n-1} + 1 \)

Write a possible explicit rule for the nth term of each sequence.

12. -4, -8, -12, -16, -20, ...
13. 5, 20, 80, 320, 1280, ...
14. -24, -19, -14, -9, -4, ...
15. 27, 18, 12, 8, 16, 3, ...
16. Sports Suppose that a basketball is dropped from a height of 3 ft. If the ball rebounds to 70% of its height after each bounce, how high will the ball reach after the 4th bounce? the 9th bounce?

Answers

1. arithmetic sequence; geometric sequence
2. diverges; converges
3. explicit formula; recursive formula
4. infinite sequence; finite sequence
5. iteration
6. -8, -7, -6, -5, -4
7. \( \frac{1}{2}, 2, \frac{9}{2}, 8, \frac{25}{2} \)
8. \( 1, \frac{3}{2}, 9, \frac{27}{4}, 81, \frac{81}{16} \)
9. 55, 53, 51, 49, 47
10. 200, 40, 8, \( \frac{8}{2^{}} \), \( \frac{8}{2^{}} \), \( \frac{8}{2^{}} \)
11. -3, 10, -29, 88, -263
12. \( a_n = -4n \)
13. \( a_n = 5(4)^{n-1} \)
14. \( a_n = 5n - 29 \)
15. \( a_n = 27\left(\frac{2}{3}\right)^{n-1} \)
16. 0.72 ft, or 8.6 in.; 0.12 ft, or 1.5 in.
12-2 Series and Summation Notation (pp. 870–877)

**EXAMPLES**

Expand \( \sum_{k=1}^{3} (-1)^{n+1} (11 - 2n) \), and evaluate.

\[
\sum_{k=1}^{3} (-1)^{n+1} (11 - 2n) = (-1)^2(11 - 2) + (-1)^3(11 - 4) + (-1)^4(11 - 6) + (-1)^5(11 - 8) + (-1)^6(11 - 10)
\]

\[= 9 - 7 + 5 - 3 + 1 = 5 \text{ Simplify.} \]

Evaluate \( \sum_{k=1}^{8} k^2 \).

Use summation formula for a quadratic series.

\[
\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}
\]

\[
\sum_{k=1}^{8} k^2 = \frac{8(8 + 1)(2 \cdot 8 + 1)}{6} = \frac{72(17)}{6} = 204
\]

12-3 Arithmetic Sequences and Series (pp. 879–887)

**EXAMPLES**

Find the 12th term for the arithmetic sequence 85, 70, 55, 40, 25, ….

Find the common difference: \( d = 70 - 85 = -15 \).

\[a_n = a_1 + (n - 1)d \text{ General rule} \]

\[a_{12} = 85 + (12 - 1)(-15) \text{ Substitute.} \]

\[= -80 \text{ Simplify.} \]

Find \( \sum_{k=1}^{18} (-2 - 33k) \).

Find the 1st and 11th terms.

\[a_1 = -2 - 33(1) = -35 \]

\[a_{11} = -2 - 33(11) = -365 \]

Find \( S_{11} \).

\[S_n = \frac{n(a_1 + a_n)}{2} \text{ Sum formula} \]

\[S_{11} = \frac{11(-35 - 365)}{2} \text{ Substitute.} \]

\[= -2200 \]

**EXERCISES**

Expand each series and evaluate.

17. \( \sum_{k=1}^{8} (-1)^k \) 

18. \( \sum_{k=1}^{5} (0.5k + 4) \)

19. \( \sum_{k=1}^{8} (-1)^{k+1}(2k - 1) \)

20. \( \sum_{k=1}^{5} \frac{5k}{k} \)

Evaluate each series.

21. \( \sum_{k=1}^{12} 5 \) 

22. \( \sum_{k=1}^{15} k^2 \) 

23. \( \sum_{k=1}^{15} k \)

24. **Finance** A household has a monthly mortgage payment of $1150. How much is paid by the household after 2 years? 15 years?

25. **Examples**

26. **Arithmetic Sequences and Series**

27. **Exercises**

28. **Answers**

29. **Savings** Kelly has $50 and receives $8 a week for allowance. He wants to save all of his money to buy a new mountain bicycle that costs $499. Write an arithmetic sequence to represent the situation. Then find whether Kelly will be able to buy the new bicycle after one year (52 weeks).

30. **Study Guide: Review 913**
EXAMPLES

Find the 8th term of the geometric sequence 6, 24, 96, 384, ….

Find the common ratio. \( r = \frac{24}{6} = 4 \)
Write a rule, and evaluate for \( n = 8 \).

\[ a_n = a_1r^{n-1} \quad \text{General rule} \]
\[ a_8 = 6(4)^{8-1} = 98,304 \]

Find the 8th term of the geometric sequence with \( a_1 = -1000 \) and \( a_8 = -40 \).

Step 1 Find the common ratio.

\[ a_8 = a_1r^{8-1} \quad \text{Use the given terms.} \]
\[ -40 = -1000r^7 \quad \text{Substitute.} \]
\[ \frac{1}{25} = r^7 \quad \text{Simplify.} \]
\[ r = \pm \frac{1}{5} \]

Step 2 Find \( a_1 \) using both possible values for \( r \).

\[ a_1 = \frac{1}{5}^4 \quad \text{or} \quad a_1 = \frac{1}{5}^6 \]
\[ a_1 = -125,000 \quad \text{or} \quad a_1 = 125,000 \]

Step 3 Write the rule and evaluate for \( a_8 \) by using both possible values for \( r \).

\[ a_n = a_1r^{n-1} \quad \text{or} \quad a_n = a_1(\frac{1}{5})^{n-1} \]
\[ a_8 = -125,000 \quad \text{or} \quad a_8 = 125,000 \quad (\frac{1}{5})^{8-1} \]
\[ a_8 = -1.6 \quad \text{a} \]
\[ a_8 = 1.6 \]

Find \( \sum_{k=1}^{7} -2(5)^{k-1} \).

Find the common ratio. \( r = \frac{\sqrt{5}}{\sqrt{5}} = -2 \)
Find \( S_n \)

\[ S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right) \quad \text{Sum formula} \]
\[ S_7 = \frac{1 - (-2)^7}{1 - (-2)} \quad \text{Substitute.} \]
\[ = \frac{1 - (128)}{3} = 129 \]

EXERCISES

Find the 8th term of each geometric sequence.

36. 40, 4, 0.4, 0.04, 0.004, …
37. \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots \)
38. \( -16, -8, -4, -2, \ldots \)
39. \( -6, 12, -24, 48, \ldots \)

Find the 9th term of the geometric sequence with the given terms.

40. \( a_1 = 24 \) and \( a_4 = 96 \)
41. \( a_1 = \frac{2}{3} \) and \( a_2 = -\frac{4}{3} \)
42. \( a_4 = 1 \) and \( a_6 = -4 \)
43. \( a_4 = 1 \) and \( a_6 = 500 \)

Find the geometric mean of each pair of numbers.

44. 10 and 2.5
45. \( \frac{5}{2} \) and 8

Find the indicated sum for each geometric series.

46. \( \sqrt{3} \) and \( \sqrt{3} \)
47. \( \frac{5}{12} \) and \( \frac{125}{108} \)

Find the indicated sum for each geometric series.

48. \( S_n \) for \( 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \)
49. \( S_n \) for \( -\frac{4}{5} + 8 - 80 + 800 + \cdots \)

50. \( \sum_{k=1}^{n} (4)^{k-1} \)
51. \( \sum_{k=1}^{n} -2(5)^{k-1} \)
52. \( \sum_{k=1}^{n} 60(-\frac{1}{2})^{k-1} \)
53. \( \sum_{k=1}^{n} 18(\frac{1}{2})^{k-1} \)

54. Depreciation A new photocopier costs $9000 and depreciates each year such that it retains only 65% of its preceding year’s value. What is the value of the photocopier after 5 years?

55. Rent A one-bedroom apartment rents for $650 a month. The rent is expected to increase by 6% per year.

a. What will be the annual rent expense on the apartment after 5 years?

b. What will be the total amount spent on rent if a person rents the apartment for the entire 5-year period?
**EXAMPLES**

Find the sum of each infinite series, if it exists.

1. 
   \[ -9261 + 441 - 21 + 1 + \cdots \]
   \[ r = \frac{-9261}{441} = \frac{1}{21} \] \hspace{1cm} Converges; \(|r| < 1 \]
   \[ S = \frac{a_1}{1 - r} \] \hspace{1cm} Sum formula
   \[ = \frac{-9261}{1 - \left(-\frac{1}{21}\right)} = -9261 \]
   \[ = \frac{194,481}{22}, \text{ or } -8840.045 \]

2. 
   \[ \sum_{k=1}^{\infty} -\frac{7}{10^k} \]
   \[ r = \frac{-7}{10} \] \hspace{1cm} Converges; \(|r| < 1 \]
   \[ S = \frac{a_1}{1 - r} \] \hspace{1cm} Sum formula
   \[ = \frac{-\frac{7}{10}}{1 - \left(-\frac{7}{10}\right)} = -\frac{7}{10} \]

**EXERCISES**

Find the sum of each infinite series, if it exists.

61. \( 2 + 5 + \cdots + (3n - 1) = \frac{n}{2}(3n + 1) \)

62. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \left( \frac{1}{8^k} \right) \]

63. \[ \sum_{k=1}^{\infty} \left( \frac{4}{3} \right)^k \]

64. 
   \[ 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 2 \]

65. 
   \[ 1 + 5 + 25 + \cdots + 5^{n-1} = \frac{n^2 - 1}{4} \]

66. 
   \[ \frac{1}{2} + \frac{1}{15} + \cdots + \frac{1}{4n^2 - 1} = \frac{n}{2n + 1} \]

67. **Recreation** A child on a swing is let go from a vertical height so that the distance that he travels in the first back-and-forth swing is exactly 9 feet.

   a. If each swing decreases the distance by 85%, write an infinite geometric series that expresses the distance that the child travels in feet.
   
   b. What is the total distance that the child in the swing travels before the swing stops?

**Answers**

56. \(-2700 + 900 - 300 + 100 + \cdots \)
57. \(-1.2 - 0.12 - 0.012 - 0.0012 + \cdots \)
58. \(-49 - 4.9 + 0.49 - 0.049 + \cdots \)
59. \[ 4 + \frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \cdots \]
60. \[ \sum_{k=1}^{\infty} \frac{9}{k^3} \]
61. \[ \sum_{k=1}^{\infty} -7\left( \frac{3}{5} \right)^k \]
62. \[ \sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{1}{8^k} \]
63. \[ \sum_{k=1}^{\infty} \left( \frac{4}{3} \right)^k \]

64. 
   \[ 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 2 \]

65. 
   \[ 1 + 5 + 25 + \cdots + 5^{n-1} = \frac{n^2 - 1}{4} \]

66. 
   \[ \frac{1}{4(1^2) - 1} = \frac{1}{2(1) + 1} = \frac{1}{3} \]

67a. \[ \sum_{k=1}^{\infty} 9(0.85)^{k-1} \]

b. 60 ft
Find the first 5 terms of each sequence.
1. \(a_n = n^2 - 4\) \(-3, 0, 5, 12, 21\)
2. \(a_1 = 48\) and \(a_n = \frac{1}{2}a_{n-1} - 8\) \(48, 16, 0, -8, -12\)

Write a possible explicit rule for the \(n\)th term of each sequence.
3. \(-4, -2, 0, 2, 4, \ldots\)
4. \(54, 18, 6, 2, \frac{2}{3}, \ldots\) \(a_n = \frac{1}{3} a_{n-1} - 1\)

Expand each series and evaluate.
5. \(\sum_{k=1}^{5} 5^k \cdot 5 + 40 + 135 + 320 = 500\)
6. \(\sum_{k=1}^{7} (-1)^{k+1}(k) 1 + 2 - 3 - 4 + 5 - 6 + 7 = 4\)

Find the 9th term of each arithmetic sequence.
7. \(-19, -13, -7, -1, \ldots\)
8. \(a_2 = 11.6\) and \(a_5 = 5\)

Find 2 missing terms in the arithmetic sequence 125, 65, \(\ldots\)

Find the indicated sum for each arithmetic series.
10. \(\sum_{k=1}^{20} 4 + 7 + 10 + 13 + \ldots\) 650
11. \(\sum_{k=1}^{12} (-9k + 8) = -606\)
12. The front row of a theater has 16 seats and each subsequent row has 2 more seats than the row that precedes it. How many seats are in the 12th row? How many seats in total are in the first 12 rows? 38 seats; 324 seats

Find the 10th term of each geometric sequence.
13. \(\frac{3}{256}, \frac{3}{16}, \frac{3}{4}, \ldots\) \(\ldots\)
14. \(a_4 = 2\) and \(a_5 = 8\) 8192
15. Find the geometric mean of 4 and 25, 10

Find the indicated sum for each geometric series.
16. \(\sum_{k=1}^{6} 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots\) 63
17. \(\sum_{k=1}^{250} \left(\frac{1}{16}\right)^{k-1} = 208.32\)
18. You invest $1000 each year in an account that pays 5% annual interest. How much is the first $1000 you invested worth after 10 full years of interest payments? How much in total do you have in your account after 10 full years? $1628.89; $13,206.79

Find the sum of each infinite geometric series, if it exists.
19. \(200 - 100 + 50 - 25 + \ldots\) \(\ldots\)
20. \(\sum_{k=1}^{n} \left(\frac{7}{8}\right)^k = 14\)

Use mathematical induction to prove \(\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \ldots + \frac{2n-1}{2} = n^2\).
21. Step 1
22. Step 2
23. Step 3

Answers

21. \(\frac{2(1) - 1}{2} = \frac{1^2}{2} = \frac{1}{2}\)
22. \(\frac{1}{2} + \cdots + \frac{2k - 1}{2} = \frac{k^2}{2}\)
23. \(\frac{1}{2} + \cdots + \frac{2k - 1}{2} + \frac{2(k + 1) - 1}{2}\)
\[= \frac{k^2}{2} + \frac{2(k + 1) - 1}{2}\]
\[= \frac{k^2 + 2k + 2 - 1}{2}\]
\[= \frac{k^2 + 2k + 1}{2} = \frac{(k + 1)^2}{2}\]
FOCUS ON SAT

When you get your SAT scores, you are given the percentile in which your scores fall. This tells you the percentage of students that scored lower than you did on the same test. You’ll see your percentile score at the national and state levels. They are usually not the same.

You may want to time yourself as you take this practice test. It should take you about 8 minutes to complete.

1. The first term of a sequence is 6, and each successive term is 3 less than twice the preceding term. What is the sum of the first four terms of the sequence?
   - (A) 27
   - (B) 30
   - (C) 51
   - (D) 57
   - (E) 123

2. The first term of a sequence is 2, and the nth term is defined to be $3n - 1$. What is the average of the 7th, 10th, and 12th terms?
   - (A) 24.5
   - (B) 28
   - (C) 29
   - (D) 32
   - (E) 84

3. The first term of an arithmetic sequence is $-5$. If the common difference is 4, what is the 7th term of the sequence?
   - (A) $-20,480$
   - (B) $-29$
   - (C) 19
   - (D) 20
   - (E) 23

4. A population of 50 grows exponentially by doubling every 4 years. After how many years will the population have 1600 members?
   - (A) 20
   - (B) 16
   - (C) 10
   - (D) 6
   - (E) 5

5. Which of the following sequences can be expressed by the rule $a_n = \frac{3n - 1}{n + 1}$?
   - (A) $3, 2, 5, 3, 2, 5, \ldots$
   - (B) $1, 2, 3, 4, 5, 6, \ldots$
   - (C) $1, 2, 3, 4, 5, 6, \ldots$
   - (D) $0, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots$
   - (E) $0, \frac{3}{2}, 3, \frac{5}{3}, 5, \ldots$

6. Which of the following sequences is a geometric sequence?
   - (A) $-7, 14, -28, 56, -112, \ldots$
   - (B) $-4, -6, -8, -10, -12, \ldots$
   - (C) $-3, 1, -3, 1, -3, \ldots$
   - (D) $4, 12, 48, 144, 576, \ldots$
   - (E) $1, 4, 9, 16, 25, \ldots$

Read each problem carefully, and make sure that you understand what the question is asking. Before marking your final answer on the answer sheet, check that your answer makes sense in the context of the question.
Short/Extended Response: Outline
Your Response

Answering short and extended response items on tests is a lot like writing essays in English class. You can use an outline to plan your response to the question. Outlines help you organize the main points and the order in which they will appear in your answer. Outlining your response will help ensure that your explanation is clearly organized and includes all necessary information.

**Outline**

1. Explain whether arithmetic or geometric.
2. Write sequence and series.
3. Find amount saved in 8th week.
4. Find total saved after 8 weeks.

An arithmetic sequence would best represent this situation because Ariana is adding $5 each week to the amount that she saves. This would be an arithmetic sequence where the first term is 40 and the common difference is 5.

The sequence for the amount saved each week is $a_n = 40 + 5(n - 1)$. The series for the total amount saved is $\sum_{k=1}^{n} [40 + 5(k - 1)]$.

The amount saved in the 8th week is $a_8 = 40 + 5(8 - 1) = 75$.

The total saved after 8 weeks is $\sum_{k=1}^{8} [40 + 5(k - 1)] = 460$. 

Clearly indicate which is the sequence and which is the series for the second step.

Include evidence to explain the answer for the first step.

Show how you found the answers for the last two steps.
Read each test item and answer the questions that follow.

**Item A**

**Short Response** Explain how to determine whether an infinite geometric series has a sum.

1. What should be included in an outline of the response for this test item?
2. Read the two different outlines below. Which outline is the most useful? Why?

**Student A**
- I. Definition of an infinite geometric series and the common ratio $r$.
- II. Definition of the sum of an infinite geometric series.
- III. Explain for which values of $r$ that a sum exists.

**Student B**
- A. Geometric series has a common ratio.
- B. Common ratio has to be less than 1.

**Item B**

**Extended Response** A pattern for stacking cereal boxes is shown at right.

- a. Explain how many boxes are in a 9-row display.
- b. If 91 boxes are to be stacked in this display, explain how many rows the display will have.

3. Read the outline below. Identify any areas that need improvement. Rewrite the outline to make it more useful.

**Outline**
- 1. Find the number of boxes if there are 9 rows.
- 2. Find the number of rows needed for 91 boxes.

**Answers to Test Items**

See answers to related problems.
CUMULATIVE ASSESSMENT, CHAPTERS 1–12

Multiple Choice

1. Which shows the series in summation notation?
   \[ \sum_{n=0}^{24} 4 + 6 + 4 + 6 + 4 \]
   \[ \sum_{n=1}^{3} (-1)^n + 5 \]
   \[ \sum_{n=1}^{3} (-1)^n + 5 \]
   \[ \sum_{n=1}^{5} (-1)^n + 5 \]

2. What is the expanded binomial?
   \[ (2x - y)^3 \]
   \[ 8x^3 - 12x^2y + 6xy^2 - y^3 \]
   \[ x^3 + 3x^2y + 3xy^2 + y^3 \]
   \[ 8x^3 + 12x^2y + 6xy^2 + y^3 \]

3. Let \( f(x) = x^3 + 2x^2 - 5x - 9 \). Which function would show \( f(x) \) reflected across the \( y \)-axis?
   \[ f(x) = -x^3 - 2x^2 + 5x + 9 \]
   \[ f(x) = -x^3 + 2x^2 + 5x + 9 \]
   \[ f(x) = 2x^3 + 4x^2 - 10x - 18 \]
   \[ f(x) = x^3 + 2x^2 - 5x - 5 \]

4. Which function shows exponential decay?
   \[ f(x) = -5x \]
   \[ f(x) = x \cdot 3.67^n \]
   \[ f(x) = 0.49(7.9)^x \]
   \[ f(x) = 5.13(0.32)^x \]

5. A ball is dropped from a height of 10 feet. On each bounce, the ball bounces 60% of the height of the previous bounce. Which expression represents the height in feet of the ball on the \( n \)th bounce?
   \[ 10(0.6)^n \]
   \[ 10(0.6)^{n-1} \]
   \[ 10 - n \]
   \[ 0.6 \]
   \[ 10(0.6)^n \]

6. Which is the graph of the inequality \( 6x + 3y \geq 9x^2 - 37 \)?

7. Gina opened a new deli. Her revenues for the first 4 weeks were \$2000, \$2400, \$2880, and \$3456. If the trend continues, which is the best estimate of Gina’s revenues in the 6th week?
   \[ \$3856 \]
   \[ \$4032 \]
   \[ \$4147 \]
   \[ \$4980 \]

8. What is the 9th term in the sequence?
   \[ a_n = \frac{1}{2}(a_{n-1}) + 4 \]
   \[ 36 \]
   \[ 68 \]

9. Find the inverse of \( f(x) = 4x - 5 \).
   \[ f^{-1}(x) = -4x + 5 \]
   \[ f^{-1}(x) = \frac{3}{4}x + 5 \]
   \[ f^{-1}(x) = \frac{x + 5}{4} \]
   \[ f^{-1}(x) = 5x - 4 \]

For Item 13, students will often forget that one of the 10 cards is also a diamond. Remind them that they must account for this by subtracting the probability of this one card from the combined probability.
10. What transformation has been applied to $f$ to get $g$?

![Graph of $f$ and $g$]

- **Horizontal compression by $\frac{1}{5}$**
- **Horizontal stretch by 5**
- **Vertical compression by $\frac{1}{3}$**
- **Vertical stretch by 5**

In Item 11, you may choose to graph, factor, complete the square, or use the Quadratic Formula to find the zeros.

11. Find the zeros of $f(x) = 2x^2 + 5x - 12$.

- **A** $-4, \frac{3}{2}$
- **B** $-2, 3$
- **C** $-\frac{3}{2}, 4$
- **D** $\frac{3}{2}, 2$

**Gridded Response**

12. Find the common ratio of the geometric sequence.

- $125, 50, 20, 8, \ldots$

**Short Response**

13. A card is drawn from a deck of 52. What is the probability of drawing a 10 or a diamond, to the nearest hundredth?

- **A** $0.25$
- **B** $0.26$
- **C** $0.27$
- **D** $0.28$

**Extended Response**

14. What is the sum of the arithmetic series?

- $\sum_{k=1}^{8} (7k - 3)$

**Answers**

15. What is the $y$-value of the point that represents the solution to the given system of equations, to the nearest hundredth?

- \[ \begin{cases} 2y - 2 = 4x \\ 6 - x = 8y \end{cases} \]

**Short-Response Rubric**

**Items 16–18**

4 Points = The student correctly draws and labels a tree diagram in part **a**, answers correctly and shows all work.

2 Points = The student correctly draws and labels a tree diagram in part **a** and answers correctly but does not show all work.

1 Point = The student does not answer correctly and does not attempt all parts of the problem.

**Extended-Response Rubric**

**Item 19**

4 Points = The student correctly draws and labels a tree diagram in part **a** and answers parts **b** and **c** correctly, but does not show all work.

2 Points = The student correctly draws and labels a tree diagram in part **a** but does not correctly answer either part **b** or **c**.

1 Point = The student draws an incomplete tree diagram in part **a** and does not correctly answer parts **b** and **c**.

0 Points = The student does not answer correctly and does not attempt all parts of the problem.
Chapter 12 Sequences and Series

Problem Solving on Location

Organizer

Objective: Choose appropriate problem-solving strategies and use them with skills from Chapters 11 and 12 to solve real-world problems.

The Hoover Dam

Reading Strategies

As students read Problem 1, check that they understand the words forecast and projection. If students are unfamiliar with these terms, ask them to reread the problem and use context clues to guess the definitions.

Using Data Ask students how they could estimate the total number of cars crossing the Hoover Dam in 2007. Multiply 16,300 by 365.

The Hoover Dam

Since its completion in 1935, the Hoover Dam has often been cited as one of the seven engineering wonders of the world. Its 6.6 million tons of concrete tame the waters of the Colorado River and form Lake Mead, the largest man-made reservoir in the United States.

Choose one or more strategies to solve each problem. For 1 and 2, use the table.

<table>
<thead>
<tr>
<th>Traffic Forecasts for the Hoover Dam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
</tr>
<tr>
<td>Number of Cars per Day</td>
</tr>
</tbody>
</table>

1. The Hoover Dam serves as a bridge between the Nevada and Arizona sides of the Colorado River. Traffic analysts project that the traffic on the dam will increase according to an arithmetic sequence. How many cars, on average, would you predict to cross the dam each day in 2017? 21,100

2. Approximately how many vehicles will cross the dam in the years 2007 through 2017, inclusive? (Hint: Assume 365 days per year.) About 75 million

3. Small trucks make up 18% of the traffic on the dam, and RVs account for another 4%. All trucks and RVs are inspected before they are allowed to cross.
   a. Assuming that other types of vehicles are not inspected, what is the probability that three consecutive vehicles arriving at a checkpoint will be inspected? 0.01
   b. What is the probability that 3 out of 5 vehicles arriving at a checkpoint will need to be inspected? 0.06

Problem-Solving Focus

Ask students what strategy they would use to solve Problem 2. Students may wish to use the strategy Solve a Simpler Problem and/or Make an Organized List. Finding the total number of cars for one year and multiplying by 11 will give students a good estimate to compare with their final answers.
Silver and Gold Mining

Nevada’s nickname is the Silver State, which is not surprising considering that more than 10 million ounces of the metal are mined in Nevada each year. Gold mining is also an essential part of Nevada’s economy. In fact, if Nevada were a nation, it would rank third in the world in gold production behind South Africa and Australia.

Choose one or more strategies to solve each problem.

1. Nevada experienced a gold-mining boom from 1981 to 1990. During this period, the number of thousands of ounces of gold mined each year can be modeled by a geometric sequence in which $a_1 = 375$ (that is, 375,000 ounces were mined in 1981) and the common ratio is 1.35. Approximately how many ounces of gold were mined in 1990? 5,585,000 oz

2. What was the total gold production in the years 1981 through 1990, inclusive? 20,471,000 oz

3. In a particular mine, the probability of discovering a profitable quantity of gold in a sector is approximately 40%. What is the probability that the miners will discover a profitable quantity of gold in 3 of the next 4 sectors? 15.36%

For 4, use the table.

4. Nevada experienced a silver boom during the 1990s. An industry analyst is collecting detailed data for all of the years from 1991 to 2000 in which silver production was outside of 1 standard deviation of the mean. For which years should she collect this data? 1991, 1992, 1995, 1997, and 1999

<table>
<thead>
<tr>
<th>Year</th>
<th>Production (million oz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1991</td>
<td>18.6</td>
</tr>
<tr>
<td>1992</td>
<td>19.7</td>
</tr>
<tr>
<td>1993</td>
<td>22.2</td>
</tr>
<tr>
<td>1994</td>
<td>22.8</td>
</tr>
<tr>
<td>1995</td>
<td>24.6</td>
</tr>
<tr>
<td>1996</td>
<td>20.7</td>
</tr>
<tr>
<td>1997</td>
<td>24.7</td>
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<tr>
<td>1998</td>
<td>21.5</td>
</tr>
<tr>
<td>1999</td>
<td>19.5</td>
</tr>
<tr>
<td>2000</td>
<td>23.2</td>
</tr>
</tbody>
</table>

Problem-Solving Focus

Although students can use formulas to solve Problems 1 and 2, they may gain a deeper understanding of these problems if they also explore them using problem-solving strategies. For example, as part of the final step of the problem-solving process, (4) Look Back, ask students which strategy might be useful in checking the answers to these problems. Possible answer: Use the strategy Make a Table. To check the answer to Problem 1, look at the gold production for 1990 in the table. To check the answer to Problem 2, add all the values in the production column of the table.